Series I, exercise 1 Let $n \in \mathbb{N}$. Prove that for any $2n$ points on the plane it is possible to join n pairs of points by segments which do not intersect.

Solution Let us draw n segments on the plane with endpoints among the given $2n$ points. If they don't intersect we are done. Assume to the contrary. Let us analyze the pair of segments AB and CD that intersect as if $ABCD$ were consecutive vertices of a convex quadrangle. Then the proper transform is to change segments into segments AC and BD which don't intersect (that is instead of the segments being the diagonals of the quadrangle they are the non-adjacent sides of it.

Now let us consider the sum of all lengths of the given segments. As the sum of the lengths of the diagonals of the convex quadrangles has to be greater than the sum of the lengths of two sides after each such exchange the total sum becomes smaller. As the set of points is finite so is the possible number of such sums. After a finite number of steps we reach the minimal sum, which finishes the proof.

Series I, exercise 2 Calculate

$$\lim_{n\to\infty}((\sqrt{3}+1)^n - \lfloor(\sqrt{3}+1)^n\rfloor),$$

where $\lfloor x\rfloor = \max\{k\in\mathbb{Z}\colon k\le x\}$.

Solution

First, observe that for any $n\in\mathbb{N}$ there are $a_n, b_n\in\mathbb{N}$ such that $(\sqrt{3}+1)^n = a_n + b_n\sqrt{3}$. From the Pascal's triangle we have

$$(\sqrt{3}+1)^n = \sum_{k=0}^{n}\binom{n}{k}(\sqrt{3})^{n-k}$$

and

$$(\sqrt{3}-1)^n = \sum_{k=0}^{n}\binom{n}{k}(\sqrt{3})^{n-k}(-1)^k.$$

Therefore, if n is even then in both sums the terms for even k are the same and for odd are opposite. Even terms of both sums are integers, so for even n $(\sqrt{3}-1)^n = a_n - b_n\sqrt{3}$. If n is odd, then for odd k the terms in both sums are the same, and so for odd n $(\sqrt{3}-1)^n = b_n\sqrt{3} - a_n$.

For even n we have

$$(\sqrt{3}+1)^n - \lfloor(\sqrt{3}+1)^n\rfloor = (\sqrt{3}+1)^n + (\sqrt{3}-1)^n - (\sqrt{3}-1)^n - \lfloor(\sqrt{3}+1)^n\rfloor$$
$$= a_n + b_n\sqrt{3} + a_n - b_n\sqrt{3} - \lfloor a_n + b_n\sqrt{3}\rfloor - (\sqrt{3}-1)^n = a_n - \lfloor b_n\sqrt{3}\rfloor - (\sqrt{3}-1)^n.$$

On the other hand, for odd n we have

$$(\sqrt{3}+1)^n - \lfloor(\sqrt{3}+1)^n\rfloor = (\sqrt{3}+1)^n - (\sqrt{3}-1)^n + (\sqrt{3}-1)^n - \lfloor(\sqrt{3}+1)^n\rfloor$$
$$= a_n + b_n\sqrt{3} + a_n - b_n\sqrt{3} - \lfloor a_n + b_n\sqrt{3}\rfloor + (\sqrt{3}-1)^n = a_n - \lfloor b_n\sqrt{3}\rfloor + (\sqrt{3}-1)^n.$$

We will now show that $a_n - \lfloor b_n\sqrt{3}\rfloor = 1$ for even n and $a_n - \lfloor b_n\sqrt{3}\rfloor = 0$ for odd n. It is easy to see that

$$a_n - \lfloor b_n\sqrt{3}\rfloor = 1 \Leftrightarrow a_n - b_n\sqrt{3} \in (0,1]$$

and

$$a_n - \lfloor b_n\sqrt{3}\rfloor = 0 \Leftrightarrow a_n - b_n\sqrt{3} \in (-1,0].$$

For $n\in\mathbb{N}$ we have

$$a_{n+2}+b_{n+2}\sqrt{3} = (a_n+b_n\sqrt{3})(1+\sqrt{3})^2 = (a_n+b_n\sqrt{3})(4+2\sqrt{3}) = (4a_n+6b_n)+(2a_n+4b_n)\sqrt{3},$$

so $a_{n+2} = 4a_n + 6b_n$ and $b_{n+2} = 2a_n + 4b_n$. We also have

$$a_{n+2} - b_{n+2}\sqrt{3} = 4a_n + 6b_n - (2a_n + 4b_n)\sqrt{3} = 4(a_n - b_n\sqrt{3}) - 2\sqrt{3}(a_n - b_n\sqrt{3})$$

$$= (a_n - b_n\sqrt{3})(4 - 2\sqrt{3}) \approx 0,5(a_n - b_n\sqrt{3}).$$

Therefore, if $a_n - b_n\sqrt{3} \in (0, 1]$, then also $a_{n+2} - b_{n+2}\sqrt{3} \in (0, 1]$, and if $a_n - b_n\sqrt{3} \in (-1, 0]$, then also $a_{n+2} - b_{n+2}\sqrt{3} \in (-1, 0]$. For $n = 1$ $a_1 - b_1\sqrt{3} = 1 - \sqrt{3} \in (-1, 0]$, so, by induction, for all n we have $a_{2n-1} - b_{2n-1}\sqrt{3} \in (-1, 0]$. For $n = 2$ $a_2 - b_2\sqrt{3} = 4 - 2\sqrt{3} \in (0, 1]$, so, by induction, for all n we have $a_{2n} - b_{2n}\sqrt{3} \in (0, 1]$. Thus, for $n \in \mathbb{N}$

$$(\sqrt{3} + 1)^{2n} = a_{2n} - \lfloor b_{2n}\sqrt{3} \rfloor - (\sqrt{3} - 1)^{2n} = 1 - (\sqrt{3} - 1)^{2n} \to 1$$

and

$$(\sqrt{3} + 1)^{2n-1} = a_{2n-1} - \lfloor b_{2n-1}\sqrt{3} \rfloor + (\sqrt{3} - 1)^{2n-1} = (\sqrt{3} - 1)^{2n} \to 0.$$

Hence the limit $\lim\limits_{n\to\infty}((\sqrt{3} + 1)^n - \lfloor(\sqrt{3} + 1)^n\rfloor)$ does not exist.

Series I, exercise 3 Find all pairs (n, k) of natural numbers which satisfy

$$\frac{1}{n} + \frac{1}{k} = \frac{3}{2018}.$$

Solution

First rewrite the equation as $2 \cdot 1009(n+k) = 3nk$, and note that 1009 is prime, so at least one of a and b must be divisible by 1009. If both n and k are divisible by 1009, say with $n = 1009q$, $k = 1009r$, then we have $2(q+r) = 3qr$. But $qr \geq q+r$ for integers $q, r \geq 2$, so at least one of q, r is 1. This leads to the solutions $q = 1$, $r = 2$ and $r = 1$, $q = 2$, corresponding to the ordered pairs $(n, k) = (1009, 2018)$ and $(n, k) = (2018, 1009)$.

In the remaining case, just one of n and k is divisible by 1009, say $n = 1009q$. This yields $2 \cdot 1009(1009q + k) = 3 \cdot 1009qk$, which can be rewritten as $2 \cdot 1009q = (3q-2)k$. Because the prime 1009 does not divide k, it must divide $3q-2$; say $3q - 2 = 1009a$. Then $1009a + 2 = 3 \cdot 336a + a + 2$ is divisible by 3, so $a = 1(mod 3)$. For $a = 1$, we get $q = 337$, $n = 1009337$, $k = 2q = 674$. For $a = 4$, we get $q = 1346$, $n = 1009 \cdot 1346$, $k = q/2 = 673$. We now show there is no solution with $a > 4$. Assuming there is one, the corresponding value of q is greater than 1346, and so the corresponding $k = \frac{2q}{3q-2}109$ is less than 673. Because k is an integer, it follows that $k \leq 672$, which implies $\frac{1}{k} \geq \frac{1}{672} \geq \frac{3}{2018}$, contradicting $\frac{1}{n} + \frac{1}{k} = \frac{3}{2018}$. Finally, along with the two ordered pairs (n, k) for which n is divisible by 1009 and k is not, we get two more ordered pairs by interchanging n and k.