# A Hartogs type theorem for constructible functions over real closed fields

Krzysztof Kurdyka (Université Savoie Mont Blanc)

joint work with Wojciech Kucharz

*XLVI AAG Lodz 26*

## Definition

Let $R$ be a real closed field. A subset $A \subset R^n$ is algebraically constructible (constructible for short) if it is a finite union of differences of algebraic sets. (Then $A$ is a semialgebraic set).

More generally, let $U \subset R^m$ and $A' \subset U$, we say that $A'$ algebraically constructible in $U$ if there exists a constructible set $A \subset R^n$ such that $A' = A \cap U$.

Let $A \subset R^m$ be a constructible set. A function (map) $f : A \to R^n$ is constructible iff its graph is an algebraically constructible subset of $R^m \times R^n$.

More generally, let $A \subset R^m$ and $B \subset R^n$ be semialgebraic sets. A function $f : A \to R^n$ is relatively constructible iff its graph is algebraically constructible in $A \times B$.

## Examples

Every rational function is constructible. Cubic root defined on $R$ is a constructible function.

Composition of constructible functions is a constructible function.

Consider the algebraic surface

$$S = \{x^3 - (1 + z^2)y^3 = 0\} \subset \mathbb{R}^3$$

Then $S$ is an analytic submanifold and the function $f : S(\mathbb{R}) \to \mathbb{R}$, defined by $f(x, y, z) = (1 + z^2)^{1/3}$, is constructible. Furthermore, $f$ is a continuous rational function on $S$ since $f(x, y, z) = x/y$ on $S$ without the $z$-axis. On the other hand, $f$ restricted to the $z$-axis is not a rational function. Thus $f$ is not hereditarily rational (regulous) but is constructible.

### Lemma (1)

*Let $A \subset R^m$ be an open semialgebraically connected set and $B \subset R^n$ be an open semialgebraic set. Let $f, g : A \to B$ be two relatively constructible continuous functions. Assume $f = g$ on $\Omega \neq \emptyset$, where $\Omega \subset A$ is an open semialgebraic set. Then $f = g$ in $A$.*

### Proof.

The set $Z = \{a \in A : f(a) = g(a)\}$ is semialgebraic, closed in $A$. Set $W = \overline{Int Z} \cap A$, since $\Omega \subset Z$ hence $W \neq \emptyset$. Let $p \in W$, then the germs of the Zariski closure of the graph of $f$ and of $g$, at the point $(p, f(p)) = (p, g(p))$ have only one component of dimension $m$. These components contain common semialgebraic germ of dimension $m$, hence they are equal. Thus $W$ is open, being also closed, it follows that $A = W$.

$\square$

## Theorem (2*)

*Let R be a real closed Archimedean field, U an open semialgebraic subset of $R^n$, and $f : U \to R$ a bounded function which is continuous semialgebraic with respect to each variable separately. Assume that there exists a nonzero polynomial P on $R^{n+1}$ such that*

$$P(x, f(x)) = 0$$

*for all $x \in U$. Then f is a semialgebraic function.*

)* Proposition 2.9 in KK, WK, Ali El-Siblani; BLMS 2021.

## Lemma (3)

*Let R be an uncountable real closed field, and $\Omega \subset R^n$ an open rectangle. Let $f : \Omega \to R$ be a function which is continuous relatively constructible with respect to each variable separately. Then there exists a nonzero polynomial P on $R^{n+1}$ such that*

$$P(x, f(x)) = 0 \text{ for all } x \in \Omega$$

In the proof of Proposition 2.4 (KK, WK, Ali El-Siblani; BLMS 2021.) replace "Nash" by "continuous relatively constructible " and apply Lemma 1.

# Main theorem

Let $R$ be a uncountable real closed Archimedean field. Let $X, Y$ be real algebraic varieties over $R$, $A \subset X$, $B \subset Y$ constructible subsets, and $f \colon A \to B$ a map. Then the following conditions are equivalent:

1. $f$ is a continuous constructible map.
2. For every irreducible Zariski closed curve $C \subset X$, the restriction $f|_{C \cap A}$ $f$ is a continuous constructible map.

## Proof.
First using Theorem (2*) and Lemma 3 we prove that the map $f$ is semialgebraic. $\qquad\square$

Then we use the following characterization of constructible sets:

### Lemma (4)

*Let $X$ be a real algebraic variety an let $S$ be a semialgebraic subset of $X$. Assume that for every irreducible Zariski closed curve $D \subset X$, the intersection $D \cap S$ is a constructible subset of $X$. Then $S$ is a constructible set.*

### Example

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as follows: $f(x, y) = 1$ if $y = e^x$ and $f(x, y) = 0$ if $y \neq e^x$. Then for each irreducible closed curve $C \subset \mathbb{R}^2$ the restriction of $f$ to $C$ is constructible but not continuous. Note that $f$ is not semialgebraic, hence is not constructible.