Symplektyczna reprezentacja klas odwzorowań typu algebraicznie skończonego na powierzchniach

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WYDZIAŁ MATEMATYKI I INFORMATYKI
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Torus homeomorphisms

- Mapping class group of an orientable surface

$$\mathrm{Mod}(\Sigma) = Homeo^{+}(\Sigma)/Homeo_0(\Sigma)$$

- For the torus T^2:

$$\mathrm{Mod}(T^2) \xrightarrow{\cong} SL(2,\mathbb{Z})$$
$$f \mapsto H_1(f) =: A$$

- The characteristic polynomial of A is $x^2 - \mathrm{tr}(A)x + 1$ with eigenvalues λ, λ^{-1}.

$\mathbb{H}^2$ isometry	elliptic	parabolic	hyperbolic
$\lvert\mathrm{tr}(A)\rvert$	0,1	2	$3, 4, \ldots$
eigenvalues λ, λ^{-1}	distinct, complex	both equal to ± 1	distinct, real
mapping class	periodic	reducible	Anosov
example	$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

- If f is reducible, it fixes a circle determined by a rational eigenvector of A.

Surfaces Σ_g of genus $g \geq 2$

• A homeomorphisms $f\colon \Sigma_g \to \Sigma_g$ induces $H_1(f) =: \Psi([f]) \in Sp(2g, \mathbb{Z})$ — the symplectic group, i.e. matrices A satisfying

$$\Omega = A^T \Omega A, \quad \text{where} \quad \Omega = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}.$$

Theorem (H. Burkhardt (1889))

The homomorphism $\Psi\colon \mathrm{Mod}(\Sigma_g) \to Sp(2g, \mathbb{Z})$ is surjective.

Theorem (Nielsen–Thurston)

Each mapping class $[f] \in \mathrm{Mod}(\Sigma_g)$ is periodic, reducible, or pseudo-Anosov. More precisely, there are four disjoint families of mapping classes:

- T1 — *periodic homeomorphisms,*
- T2 — *reducible non-periodic homeomorphisms without pseudo-Anosov pieces,*
- T3 — *reducible homeomorphisms with a pseudo-Anosov piece,*
- T4 — *pseudo-Anosov maps.*

Algebraically finite type

• Nielsen in 1944 [12] called mapping classes from T1 and T2 of *algebraically finite type* (a.f.t.). By definition such a class is represented by f that is either periodic or reducible and taking invariant system $\{C_i\}$ of circles f is periodic on $\Sigma_g \setminus \bigcup C_i$.

• Nielsen showed that classes of a.f.t. are *quasi-unipotent*, i.e. the spectrum of $H_1(f)$ consists only of roots of unity.

Conjecture (Nielsen 1944 [12])

All quasi-unipotent mapping classes are of algebraically finite type.

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Theorem (Thurston)

For $g \geq 2$ there exists a pseudo-Anosov diffeomorphism $f : \Sigma_g \to \Sigma_g$ such that $H_1(f) = \mathrm{id}$ on $H_1(\Sigma_g)$.

Theorem (da Rocha 1985 [2])

For a mapping class of Σ_g, $g \geq 2$, TFAE:
1) it is of algebraically finite type 2) it contains a Morse–Smale diffeomorphism.

Symplectic representation of pA and a.f.t

Proposition

The restriction $\Psi\colon T4 \to Sp(2g, \mathbb{Z})$ is surjective, i.e. every symplectic transformation can be realized by a pseudo-Anosov map.

Proof idea

For $A \in Sp(2g, \mathbb{Z})$ take any f such that $H_1(f) = A$ and let h be a pseudo-Anosov map in the Torelli group $\mathcal{I}_g = \mathrm{Ker}(\Psi\colon \mathrm{Mod}(\Sigma_g) \to Sp(2g, \mathbb{Z}))$. Then for a sufficient large n, $h^n \circ f$ is pseudo-Anosov.

Under very specific conditions, one can detect from $\Psi([f])$ that $[f]$ is pseudo-Anosov (Casson–Bleiler [1]).

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Problem

Is the restriction $\Psi\colon T1 \cup T2 \to Sp(2g, \mathbb{Z}) \cap QU(2g, \mathbb{Z})$ surjective?
If not, what is the image?

Virtual homological eigenvalues

• If $\tilde{f}$ is a lift of f for a finite covering $\widetilde{\Sigma} \to \Sigma$, then the eigenvalues (spectral radius) of $H_1(\tilde{f})$ are called a *virtual eigenvalues* (spectral radius) of f.

Proposition

If f has no pseudo-Anosov piece (so is of a.f.t.), then its virtual homological eigenvalues are all roots of unity.

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- Let $\lambda_f > 1$ be the largest stretch factor of pseudo-Anosov pieces of $f \in T3 \cup T4$. If $f \in T1 \cup T2$, then $\lambda_f = 1$.
- $N^\infty(f) = \limsup \sqrt[n]{N(f^n)}$ — the asymptotic Nielsen number.
- $h(f)$ — the topological entropy of f.

Theorem (Boju Jiang)

$$N^\infty(f) = \lambda_f \quad \textit{and} \quad h(f) = \log \lambda_f .$$

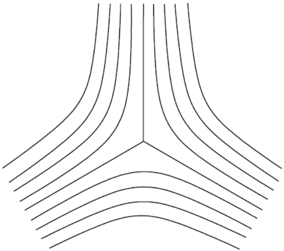
McMullen gap theorem

Theorem (McMullen 2013 [11])

Let f be a pseudo-Anosov map with stretch factor $\lambda_f > 1$.

(1) *If the invariant foliations of f have no singularities of odd prong numbers, then λ_f is a virtual homological eigenvalue of f.*

(2) *Otherwise, there exists some constant $1 < r < \lambda_f$, depending only on f, such that every virtual homological eigenvalue μ of f satisfies $|\mu| < r$.*



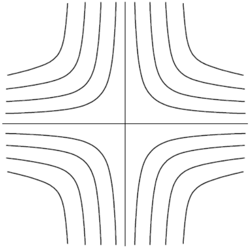


Figure: A foliation at a three-pronged singular point (left) and at a four-pronged singular point (right), [4].

Virtual homological eigenvalues

Theorem (Yin Liu 2019 [10], partially T. Koberda 2010; cf. A. Hadari 2020 [8] for surfaces with boundary)

McMullen conjecture is true:
A surface automorphism has a pseudo-Anosov piece if and only if it has a virtual homological eigenvalue outside the complex unit circle.

- Mapping torus of f:

$$M_f = \frac{\Sigma \times \mathbb{R}}{(x, r+1) \sim (f(x), r)}$$

- The fibration $M_f \to \mathbb{R}/\mathbb{Z}$ defines distinguished class $\phi_f \in H^1(M_f; \mathbb{Z})$.

- For a finite covering $\widetilde{M} \to M_f$ we can identify $\widetilde{M}$ with a mapping torus $M_{\tilde{f}}$ of some lift $\tilde{f}$ of f ($\widetilde{\Sigma}$ possibly disconnected), and so define the distinguished class $\phi_{\tilde{f}} \in H^1(M_{\tilde{f}}, \mathbb{Z})$.

Fibered cones

- The Thurston norm of $\phi \in H^1(M_f, \mathbb{Z})$:

$$||\phi||_{Th} := \min_S \sum_i \max(0, -\chi(S_i)),$$

over all embedded surfaces S Poincaré dual to ϕ with connected components S_i.

- Let f be a pseudo-Anosov map.
- The unit ball B of $||\cdot||_{Th}$ around 0 in $H^1(M_f, \mathbb{R})$ is a compact convex polyhedron.
- If ϕ is fibered, Thurston showed that it is contained in the so-called *fibered cone* over an open codimension-one face of B, in which the integral classes are all fibered.

Theorem (Yin Liu 2019 [10])

If f is a pseudo-Anosov map, then for any natural number n, there exists a finite regular cover $\widetilde{M}$ of M_f, such that the fibered cone of the distinguished class of $\widetilde{M}$ has at least n distinct deck transformation orbits of codimension-one faces.

A specific criterion for proving McMullen conjecture relies on the Mahler measure of the multivariable Alexander polynomial $\Delta^{\#}_{M_f} \in \mathbb{Z}H_1(M_f;\mathbb{Z})_{free}$.

Homotopy invariants of dynamics

- $L(f)$ is the Lefschetz number of f.
- **Dold coefficients**

$$a_n(f) = \frac{1}{n}\sum_{k|n}\mu\left(\frac{n}{k}\right)L(f^k)\in\mathbb{Z}, \quad \text{so} \quad L(f^n)=\sum_{k|n}ka_k(f),$$

and the set of **algebraic periods** of f:

$$\mathcal{A}P(f) := \{n\in\mathbb{N} : \ a_n(f)\neq 0\}$$

- $P_n(f)$ — the set of n-periodic points of f.

Proposition (see [9])

Let f be a transversal map (e.g. a Morse–Smale diffeomorphsim), i.e. $x\in \mathrm{Fix}(f^n)$, implies $1\notin \mathrm{Spec}(D^n_x f)$.
If $a_n(f)\neq 0$, then $\begin{cases} |P_n(f)|\geq |a_n(f)| & \text{if } n \text{ is odd,}\\ |P_n(f)\cup P_{n/2}(f)|\geq |a_n(f)| & \text{if } n \text{ is even.}\end{cases}$

Construction of homeomorphisms of a.f.t.

G. Graff, WM, ŁPM 2025 [6]: classification of possible Dold coefficients of quasi-unipotent surface homeomorphisms.

Theorem (G. Graff, WM, ŁPM, A. Myszkowski 2025 [5])

If $\mathcal{A} \subset \mathbb{N}$ is finite, then there exists a Morse–Smale diffeomorphism $f : \Sigma_g \to \Sigma_g$ such that

$$\mathcal{AP}(f) = \mathcal{A} \qquad \text{and} \qquad g = \pm 1 + \sum_{n \in \mathcal{A}} n.$$

In fact, the matrix of $H_1(f)$ is

$$\bigoplus_{n \in \mathcal{A}'} C_n \oplus C_n,$$

where $\mathcal{A}' = (\mathcal{A} \setminus \{1\}) \cup (\{1\} \setminus \mathcal{A})$ and C_n is the permutation matrix for a cycle of length n. Therefore

$$\operatorname{ord}(H_1(f)) = \operatorname{lcm}(n \in \mathcal{A}).$$

Quasi-unipotent homeomorphisms

Proposition

For a homeomorphism $f : \Sigma_g \to \Sigma_g$ TFAE:
(1) $(L(f^n))$ is bounded, (2) $\mathcal{A}P(f)$ is finite, (3) $H_1(f)$ is quasi-unipotent.

Describe the spectrum of $H_1(f)$ by a sequence (r_k) such that it has r_k times all primitive roots of unity of degree k.

Theorem (G. Graff, WM, ŁPM [6])

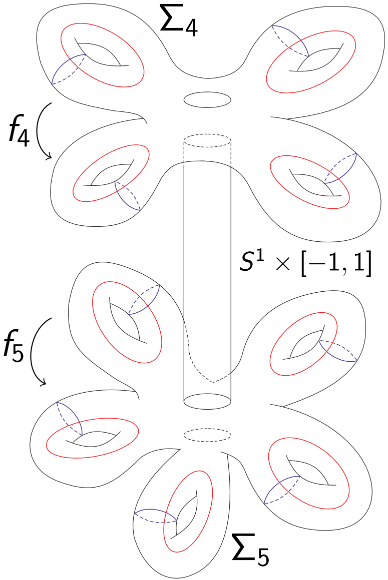
For a sequence (a_n) with finitely many non-zero terms there exists a quasi-unipotent orientation-preserving homeomorphism $f : \Sigma_g \to \Sigma_g$ such that $a_n(f) = a_n$ if and only if

$$r_1 = 2 - \sum_n a_n \geq 0, \quad r_2 = -\sum_n a_{2n} \geq 0, \ \ldots, \ r_k = -\sum_n a_{kn} \geq 0,$$

r_1 and r_2 are even and

$$2g = \sum_k r_k \varphi(k) = 2 - \sum_n n a_n(f).$$

Construction of homeomorphisms of a.f.t.



Orders of torsion elements

Theorem ($84(g-1)$ theorem and $4g+2$ theorem, see [4])

Let $g \geq 2$. The order of any finite subgroup of $\mathrm{Mod}(\Sigma_g)$ is at most $84(g-1)$ and the order of any torsion element is at most $4g+2$.

Question

What is the upper bound for torsion elements in $\Psi(T2)$?

Theorem (Ł. Michalak)

For any g there exists an algebraically finite type mapping class $[f]$ of Σ_g such that $\Psi([f]) = H_1(f)$ has finite order that is asymptotically

$$e^{(1+o(1))\sqrt{g \log g}}.$$

For a fixed n consider $\mathcal{A} = \{p : p \leq n \text{ and } p \text{ is prime}\}$. Then

$$\frac{n^2}{\log n^2} \sim \pm 1 + \sum_{p \leq n} p = g = \frac{g \log g}{\log g} \geq \frac{g \log g}{\log(g \log g)}, \quad \text{so} \quad n \gg \sqrt{g \log g}.$$

$$\mathrm{ord}(H_1(f)) = \mathrm{lcm}(p \in \mathcal{A}) = \prod_{p \leq n} p =: n\# \geq (2{,}076)^n \gg (2{,}076)^{\sqrt{g \log g}}.$$

Orders of torsion elements

Definition

Landau's function $\mathcal{G}(n)$ = the largest order of an element of the symmetric group S_n = the largest lcm of a partition of n.

Theorem (Landau 1902)

$$\lim_{n\to\infty} \frac{\ln \mathcal{G}(n)}{\sqrt{n \ln n}} = 1.$$

Definition

$\mathcal{H}(n)$ = the largest order of an element of finite order of $Gl(n,\mathbb{Z})$, $Sp(2\lfloor \frac{n}{2} \rfloor, \mathbb{Z})$ = the largest lcm of $(x_1, \ldots, x_k)$ such that $n = \sum_i \varphi(x_i)$.

$S_n \hookrightarrow Gl(n,\mathbb{Z})$, so $\mathcal{G}(n) \leq \mathcal{H}(n)$.

Theorem (Levitt–Nicolas 1998)

$$\lim_{n\to\infty} \frac{\ln \mathcal{H}(n)}{\sqrt{n \ln n}} = 1.$$

Characteristic polynomial for periodic mapping classes

Theorem (Nielsen 1937)

Let $f\colon \Sigma_g \to \Sigma_g$ be an n-periodic homeomorphism. Then the action of $\mathbb{Z}_{/n} =< f >$ leads to a branched covering $q\colon \Sigma_g \to \Sigma_{g_{/<f>}} =: \Sigma_h$ with u branch points $x_1, \ldots, x_u$ and $m_i = \# q^{-1}(x_i)$. Moreover, the characteristic polynomial of $H_1(f)$ is equal to

$$\chi_{H_1(f)}(x) = \frac{(x-1)^2(x^n-1)^{2h+u-2}}{(x^{m_1}-1)(x^{m_2}-1)\cdots(x^{m_u}-1)}$$

The ramification indices $\pi_i = \{k^i_j\}_{j=1}^{m_i}$ of points over x_i form a partition of n. The tupple:

$$(g, h, n, u; \pi_1, \ldots, \pi_u)$$

is called a *branch data*.

Problem (Hurwitz)

Which (candidate) branch data $(g, h, n, u; \pi_1, \ldots, \pi_u)$ is realizable by a branch covering?

Characteristic polynomial for T2 mapping classes

Let $f\colon \Sigma_g \to \Sigma_g$ be a reducible homeomorphism of class T2 with a canonical reduction system C_i of disjoint circles. Then f permutes components of $\Sigma_g \setminus \bigcup_i C_i$ and for each component S_l there is a minimal β_l such that f^{β_l} maps S_l onto itself and is n_l-periodic with a branch datum $m_{l1}, \ldots, m_{lu_l}$.

Theorem (Nielsen 1944 [12])

The characteristic polynomial of $H_1(f)$ for f as above is equal to

$$\chi_{H_1(f)}(x) = (x-1)^2 \prod_l \frac{(x^{\beta_l n_l} - 1)^{2h_l + u_l - 2}}{(x^{\beta_l m_{l1}} - 1)(x^{\beta_l m_{l2}} - 1)\cdots(x^{\beta_l m_{lu_l}} - 1)}.$$

where the product is taken over orbits of the action of f on connected components of $\Sigma_g \setminus \bigcup_i C_i$.

Characteristic polynomial for T2 mapping classes

The classification of finite group actions on a surface of low genus g was done by

- S. Broughton 1991: $g = 2, 3$,
- O. Bogopolski 1996: $g = 4$,
- A. Kuribayashi and H. Kimura : $g = 5$,
- J. Karabáš, R. Nedela and M. Skyvová 2024: $g \leq 9$ (computer computations)

Example: Orders of torsion elements in $\mathrm{Mod}(\Sigma_2)$: $1, 2, 3, 4, 5, 6, 8, 10 = 4g + 2$.

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Example: Orders of torsion elements in $\mathrm{Mod}(\Sigma_2)$: $1, 2, 3, 4, 5, 6, 8, 10 = 4g + 2$.
However, elements of T2 are of infinite order.
One can get $f \in T2$ for Σ_2 such that $\mathrm{ord}(\Psi([f])) = 12$, but the characteristic polynomial of $\Psi([f])$ is the product $\varphi_3(x)\varphi_4(x)$ of cyclotomic polynomials.

Conjecture - L. Michalak 2025

The 12th cyclotomic polynomial $\varphi_{12}(x) = x^4 - x^2 + 1$ cannot be realized as the characteristic polynomial of the symplectic transformation induced by a mapping class of algebraically finite type.
Confirmed: (T. Koberda & L. M. 2025): cannot be realized as the characteristic polynomial of the symplectic transformation induced by a mapping class $\in T1 \cup T2 \cup T3$.

Thank you!

Dziękuję!

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