A qualitative approach to hydrodynamics

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Navier–Stokes Equations

The main subject of the talk is the famous **Navier–Stokes system**

$$\dot{\boldsymbol{V}} + (\boldsymbol{V}, \nabla)\, \boldsymbol{V} - \nu \Delta \boldsymbol{V} + \operatorname{grad} P = \boldsymbol{F}, \quad \operatorname{div} \boldsymbol{V} = 0.$$

It is a system for a *vector field* $\boldsymbol{V} : \mathbb{R}_{\geq 0} \times U \longmapsto \mathbb{R}^d$ (for a domain $U \subset \mathbb{R}^d$) and for the *pressure* $P : \mathbb{R}_{\geq 0} \times U \longmapsto \mathbb{R}$; $\nu \geq 0$ is the *viscosity coefficient*. Here $\boldsymbol{F}$ is the *external force* field.

The N–S system is a generalization of the **Euler system**:

$$\dot{\boldsymbol{V}} + (\boldsymbol{V}, \nabla)\, \boldsymbol{V} = 0;$$

it describes evolution of the velocity field of a cloud of free particles. So, we call the term $(\boldsymbol{V}, \nabla)\, \boldsymbol{V}$ the *Euler term*. The term $\nu \Delta \boldsymbol{V}$ is associated with the diffusion of the particles and we call it the *diffusion term*. The term $\operatorname{grad} P$ is called the *pressure term*.

Under the assumption $\boldsymbol{F} \equiv 0$ we can scale the vector field $\boldsymbol{V}$ and other ingredients such that the viscosity coefficients becomes normalized to 1. Therefore we assume

$$\nu = 1;$$

moreover, $d = 2$ or $d = 3$.

There are many books and monographs devoted to the Navier–Stokes system; in the references list I present those which I have somehow looked into. In [AK] the motion of a fluid is associated with a suitable (infinite dimensional) group of diffeomorphisms of U. In [CF] one finds main tools and results related with the N–S system. In [DR] a wide list of examples of exact solutions is presented; also many examples are given in [Jos]. In [Jos] methods of bifurcation theory are used: Andronov–Hopf bifurcation and its generalizations (compare also [RT]). In [LK] attractors for some N–S flows are exhibited and studied. In [VF] the hydrodynamics of fluid is studied from the statistical point of view (as evolution of some measure). The physical foundations are given in [LL]. In [Hen] a geometric approach combined with bifurcations is developed.

Recall also that there are many results about the existence and properties of solutions of the Navier–Stokes systems, especially in the 2–dimensional case. Those results rely on methods from functional analysis. I refer the reader to the classical works of J. Leray [Ler] and E. Hopf [Hopf] and to more recent papers [KN, FK] (for example).

But the 3–dimensional case is still not solved and the corresponding problem is included into the list of millennium problems, see [Fef, Kar].

Recall that one has to solve the **Initial Value Problem** (IVP) for the Navier–Stokes system:

$$\dot{\boldsymbol{V}} = \Delta \boldsymbol{V} - (\boldsymbol{V}, \nabla)\, \boldsymbol{V} - \operatorname{grad} P + \boldsymbol{F}, \quad \operatorname{div} \boldsymbol{V} = 0, \;\; \boldsymbol{V}\,(0; x) = \boldsymbol{V}_0\,(x)\,.$$

Recall that this system is considered in the domain $\mathbb{R}_{\geq 0} \times U$, where $U \subset \mathbb{R}^d$ is an open domain.

As in other evolutionary equations one assumes the homogeneous (or Dirichlet) boundary condition:

$$\boldsymbol{V}|_{\partial U} \equiv 0.$$

There arises a problem with the pressure term $\operatorname{grad} P$; it does not appear in the initial condition and in the boundary condition; thus $\operatorname{div} \operatorname{grad} P$ can be nonzero. Also the Euler term $(\boldsymbol{V}, \nabla)\, \boldsymbol{V}$ can have nonzero divergence. So one has to separate somehow the divergence free part in the first group of d N–S equations.

The authors do it by combining the above vanishing restriction with the condition $\operatorname{div} \boldsymbol{V} = 0$ via the classical Gauss formula $\int_U \operatorname{div} \boldsymbol{W} = \int_{\partial U} (\boldsymbol{n}, \boldsymbol{W})\,$.

In the 2–dimensional case this problem was solved by introducing so-called Stokes operator $\mathcal{A} = -\mathcal{P}\Delta$, where the Leray's projector $\mathcal{P}$ has image in the closure of the space of divergence free vector fields vanishing at the boundary, and proving the convergence of so called Galerkin approximations.

It has turned out that this method does not work in the 3–dimensional case. Plausibly, the main problem is with the basis of some Hilbert space; but I did dot study this subject in detail.

But I have found another flaw in the above approach. In a below example I study an IVP with $U = (0, \pi) \times (0, \pi)$ with the initial divergence free vector field vanishing at the boundary and such that the diffusion term is dominant. I have skipped other terms assuming that the viscosity coefficient ν is large.

The solution is quite explicit, but the vector field does not vanish at the boundary for $t > 0$; it is only tangent to ∂U.

This is physically sensible, because the diffusion can force the fluid to move along the boundary.

Now I will present my approach which is novel and seems to be physically correct.

I realized that the core of the problem is to represent each vector field in the first d N–S equations as a sum of two other vector fields: a divergence free field and a field whose divergence is easy to compute.

Using the vector analysis, it is easy to describe the divergence free vector fields, at least for low dimensions: $d = 2$ and $d = 3$. If the domain $U \subset \mathbb{R}^d$ is contractible, then such vector field is Hamiltonian, $\boldsymbol{X}_H$ (if $d = 2$), or it is a curl curl $\boldsymbol{H}$ of another vector field $\boldsymbol{H}$ (if $d = 3$).

There are many candidates for another summand. Simplest from the mathematical point of view are vector fields parallel to one of the euclidean coordinate axes, e.g., $g\,(x) \cdot \partial_d$; then div $(g \cdot \partial_d) = \partial_d g$. The so-called *Euclidean splitting* is following:

$$\begin{aligned} \mathsf{Vect} &= \mathsf{Ham} + \mathsf{Func} \cdot \partial_2, \quad (d = 2)\,, \\ \mathsf{Vect} &= \mathsf{Curl} + \mathsf{Func} \cdot \partial_3, \quad (d = 3)\,. \end{aligned}$$

Here Vect is the space of vector fields, Func is the space of functions and Ham and Curl are naturally defined. The intersections of the two subspaces is nontrivial, although easy to describe.

Another natural choice of the second summand consist of vector fields parallel to the radius vector (or the Euler vector field): $g\,(r,\theta)\cdot\partial_r = \frac{g}{r}\boldsymbol{E}$, where $\boldsymbol{E} = \sum x_j \partial_j$. Then we have the *polar splitting* and the *spherical splitting*

$$\begin{aligned} \text{Vect} &= \text{Ham} + \text{Func}\cdot\partial_r, \quad (d=2)\,, \\ \text{Vect} &= \text{Curl} + \text{Func}\cdot\partial_r, \quad (d=3)\,. \end{aligned}$$

The latter splitting seems to be more natural from the physical point of view. Recall that the divergence measures the decompression (or compression) of the fluid (or the gas) and the second summand says in which direction this decompression acts. Moreover, the additional problems associated with the intersection of two subspaces are easier than in the case of the Euclidean splitting.

Let me present the elimination of the pressure for $d=2$ using the Euclidean splitting and the polar splitting. To this aim I use three facts:

(i) representation of $\boldsymbol{F}$ as $\boldsymbol{X}_K + L\cdot\partial_2$ (or as $\boldsymbol{X}_K + L\cdot\partial_r$) for some functions K and L;

(ii) representation of grad P as $\boldsymbol{X}_Q + R\cdot\partial_2$ (or as $\boldsymbol{X}_Q + R\cdot\partial_r$) for some functions Q and R;

(iii) vanishing of the curl of the gradient, $\nabla \times \nabla = 0$;

(iv) vanishing of the divergence of the both sides of the first group of the N–S equations.

Therefore I express the functions R (first) and Q (next) via explicit formulas involving H from $\boldsymbol{V} = \boldsymbol{X}_H$, i.e., in then Euler term. These formulas involve partial differentiation operators, the inverse of the Laplacian Δ^{-1}, and the inverse of the partial derivative: ∂_2^{-1} or ∂_r^{-1}, respectively.

The inverse of the Laplacian is understood like inverses of other elliptic operators, i.e., Δ acts on functions vanishing on ∂U with the spectrum in $\mathbb{R}_{<0}$.

The inverses of the partial derivatives are more subtle. For instance, in the case of Euclidean splitting, ∂_2^{-1} means an integration operation $\int^{x_2}$. But, to define it properly, one should determine the constant of integration; e.g., the lower limit. Therefore I get an evolutionary equation of the form

$$\dot{H} = \Delta H - G - S + K,$$

where $G = G\,(t; x, y)$ is the 'Hamiltonian' of the divergence free part of the Euler term.

But above the functions G and Q are not unique, because their definitions involve the integration operator ∂_3^{-1}. So the right hand side is defined modulo an additive constant $C(t; x_1)$. Plausibly, that constant should be fixed by the physics of the problem.

In the case of polar splitting the problem of the integration constant disappears and I put $\partial_r^{-1} = \int_0^r$.

In the case $d = 3$, I use the Euclidean splitting to eliminate the pressure and arrive at an equation of the form

$$\dot{\boldsymbol{H}} = \Delta \boldsymbol{H} - \boldsymbol{G} - \boldsymbol{Q} + \boldsymbol{K} + \operatorname{grad} m,$$

where $\boldsymbol{H}$, $\boldsymbol{G}$, $\boldsymbol{Q}$ $\boldsymbol{K}$ are vector fields such that $\boldsymbol{V} = \operatorname{curl} \boldsymbol{H}$, etc., and $m = m(x)$ is an arbitrary function due to the non–uniqueness of the thesis of the corresponding Poincaré lemma. Here also the operators ∂_3^{-1} and Δ^{-1} are used, but the latter in a more subtle way.

By a suitable choice of the 'gauge' $m(x)$, I am able to reduce the above system of three equation to a system of two equations.

$$\dot{\boldsymbol{h}}_1 = \Delta \boldsymbol{h}_1 - g_1 - q_1 + k_1 - \partial_3^{-1} \partial_1 (\Delta h_3 - k_3)$$

$$\dot{\boldsymbol{h}}_2 = \Delta \boldsymbol{h}_2 - g_2 - q_2 + k_2 - \partial_3^{-1}\partial_2(\Delta h_3 - k_3),$$

where $h_3(x)$ does not depend on t.

Like in the 2–dimensinal case the right-hand sides are defined modulo two not determined constants of integration $C_{1,2}\,(t; x_1, x_2)$.

But working with vector fields in the case of polar splitting has turned out complicated. I decided to leave the formalism of vector fields and use the language of differential forms, which is more suitable when changing the system of coordinates. Therefore, I have worked out corresponding tools for the case of nonlinear coordinate system (y_1, y_2, y_3) , in which the metric tensor is diagonal, $\mathrm{d}s^2 = c_1^2 \mathrm{d}y_1^2 + c_2^2 \mathrm{d}y_2^2 + c_3^2 \mathrm{d}y_3^2$.

Using such objects like the Hodge star operation and the Laplace–deRham operator, acting on the space $\Omega = \oplus \Omega^j$ of differential forms, I succeeded to eliminate the gradient of the pressure and arrive at the following analogue of the previous system:

$$\dot{\eta}^1 = -\Delta_{\mathrm{d}}^{(1)} \eta^1 - \gamma^1 - \sigma^1 + \kappa + \mathrm{d}m,$$

where $\eta^1, \gamma^1, \sigma^1$ are 1–forms corresponding to the vector fields $\boldsymbol{H}, \boldsymbol{G}, \boldsymbol{S}$, $m = m(y)$ is an arbitrary function and $\Delta_{\mathrm{d}}^{(1)}$ is the Laplace–deRham operator acting on 1–forms. Next, I reduce the latter system to a system of two equations with two integration constants $C_{1,2}(t; y_1, y_2)$. These constants vanish in the case of spherical coordinates.

In relation with the above I allow myself to state the following proposition, not conjecture.

Suggestion. *A correct formulation of the Initial Value Problem for the Navier–Stokes system in $U \subset \mathbb{R}^d$, $d = 2, 3$, should contain:*

the initial velocity distribution $\boldsymbol{V}_0(x)$ tangent to ∂U and

the choice of the splitting of the space Vect *of vector fields. Namely, the divergence nontrivial part should be of the form*

$$\mathsf{Func} \cdot \boldsymbol{D},$$

where $\boldsymbol{D}(x)$ is a fixed vector field of simple form (constant or linear).

Moreover, the divergence free solution $\boldsymbol{V}(t; x)$ should be sliding along the boundary.

Of course, one can imagine more complex situations. For example, the vector field $\boldsymbol{D}$ could be chosen non-integrable or even with chaotic behavior (like in the case of Lorentz vector field). Next, the splittings of the pressure term and of the Euler term could be chosen differently. I think that such complications are not relevant from the physical point of view. They could be considered only as perturbations of a fixed splitting.

The traditional way to eliminate the pressure

The traditional approach to the Navier–Stokes equations, moreover in any dimension, relies on the assumption of the Dirichlet type boundary condition

$$\boldsymbol{V}|_{\partial U} \equiv 0.$$

Then the following splitting of the space Vect of vector fields in the domain $U \subset \mathbb{R}^d$ us used:

$$\text{Vect} = \text{ker div} \oplus (\text{ker div})^\perp.$$

Here the symbol $\perp$ denotes the orthogonal complement in the corresponding Hilbert space.

It turns out that:

under the above vanishing assumption grad P *lies in* (ker div)$^\perp$.

Indeed, the Gauss formula

$$\int_U \{(\boldsymbol{V}, \nabla P) + P \cdot \text{div}\, \boldsymbol{V}\}\, \mathrm{d}^d x = \int_U \text{div}\, (P \cdot \boldsymbol{V})\, \mathrm{d}^d x = \int_{\partial U} (\boldsymbol{n}, P \cdot \boldsymbol{V})\, \mathrm{d}^{d-1}\sigma$$

justifies this statement; above $\boldsymbol{n}$ is the unit vector orthogonal to ∂U and $\mathrm{d}^{d-1}\sigma$ is the Riemann measure on ∂U.

In [CF] it is stated that:

'*Applying Leray's projector $\mathcal{P}$ to (5.1) we obtain for smooth functions $\boldsymbol{V}(t;x)$, $P(t;x)$ satisfying (5.1), (5.2) that*

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} + \nu\mathcal{A}\boldsymbol{V} + B(\boldsymbol{V},\boldsymbol{V}) = \mathcal{P}\boldsymbol{F}$$

where $\mathcal{A}$ is the Stoker operator and

$$B(\boldsymbol{V},\boldsymbol{V}) = \mathcal{P}(\boldsymbol{V}\cdot\nabla\boldsymbol{V}).$$

The procedure of applying $\mathcal{P}$ eliminates the pressure from the equations.'

(Above $\mathcal{P}$ is the orthogonal projection from $L^2(U)^d$ to $\mathcal{H} = \text{closure}(\mathcal{V})$, where $\mathcal{V} = \left\{\mathbf{\Phi} \in C_0^\infty\left(\overline{U}\right)^d : \operatorname{div}\mathbf{\Phi} = 0\right\} \subset L^2(U)^d$, and the Stokes operator $\mathcal{A} = -\mathcal{P}\Delta$. The cited equations (5.1) and (5.2) are the Navier–Stokes equations; see also Introduction. I have also modified the notations.)

It turns out that this property cannot be used in the Navier–Stokes system. Namely, I claim that:

the Dirichlet boundary condition cannot hold for general solutions of the N–S system.

Example Let $U = (0, \pi) \times (0, \pi)$. We assume that the viscosity ν is large, so that the effects of the Euler term, the pressure and the external force can be neglected.

Moreover, assume the initial condition such that the vector field is Hamiltonian and vanishes at ∂U; so the whole boundary consists of critical points of the Hamiltonian H_0. Namely, I solve the following IVP for the heat equation:

$$\begin{aligned} \dot{H} &= \Delta H, \; H(0, x) = H_0(x) \\ H_0 &= \sin^2(x_1) \cdot \sin^2(x_2) \,. \end{aligned}$$

The orthogonal basis in the space $L^2(U)$, consisting of eigenfunctions of Δ, is following:

$$f_{m,n} = (2/\pi) \cdot \sin(m x_1) \cdot \sin(n x_2) \,.$$

The corresponding eigenvalues are

$$-\left(m^2+n^2\right).$$

Using this we find that

$$\begin{aligned} H_0 &= \frac{64}{\pi^2}\sum_{k,l\geq 0} a_k a_l \cdot \sin\left(\left(2k+1\right)x_1\right)\cdot \sin\left(\left(2l+1\right)x_2\right), \\ a_k &= \left[\left(2k+3\right)\left(2k+1\right)\left(2k-1\right)\right]^{-1}, \end{aligned}$$

with the eigenvalues $\lambda_{k,l} = -(2k+1)^2 - (2l+1)^2$, and the solution to the IVP is following:

$$\begin{aligned} H\left(t;x\right) &= \frac{64}{\pi^2}\sum_{k,l\geq 0} a_k a_k \exp\left(\lambda_{k,l}t\right)\sin\left(\left(2k+1\right)x_1\right)\cdot \sin\left(\left(2l+1\right)x_2\right), \\ \lambda_{k,l} &= -\left[\left(2k+1\right)^2+\left(2l+1\right)^2\right]. \end{aligned}$$

Of course, this function vanishes at the boundary of U (the series is absolutely convergent), which means that the flow defined by the vector field $\boldsymbol{X}_H$ is tangent to ∂U. But ∇H does not vanish identically on ∂U.

Indeed, consider this gradient at a generic point on the segment $\{x_1 = 0\}$, We have

$$\partial_1 H\left(t; 0, x_2\right) = \frac{64}{\pi^2} \cdot \left\{ \sum_{k \geq 0} \frac{\mathrm{e}^{-(2k+1)^2 t}}{(2k+3)(2k-1)} \right\} \cdot \left\{ \sum_{l \geq 0} a_l \mathrm{e}^{-(2l+1)^2 t} \sin\left((2l+1) x_2\right) \right\}.$$

It is easy to check that the first sum in the above formula vanishes for $t = 0$. But, as $t > 0$ becomes large, the first term, i.e., with $k = 0$, dominates and is nonzero.

The possible physical interpretation of this phenomenon is that the diffusion plays a great role. It does not allow the fluid to be stationary at the boundary. In order to vanish the velocity at ∂U one has to introduce a corresponding term in the external force.

From the dynamical point of view we can say that the subspace $\mathcal{H}$, the image of the Leray's projector $\mathcal{P}$, is not invariant with respect to the Navier–Stokes dynamics. The two conditions, vanishing of the divergence and vanishing of the field at the boundary, are not compatible for regular vector fields.

Euclidean splitting for $d = 2$

1. Our approach (probably novel) relies on the following splitting of the space of vector fields in $\mathbb{R}^2$:

$$\mathsf{Vect} \simeq \mathsf{Ham} + \mathsf{Func} \cdot \partial_2.$$

Recall that

$$\boldsymbol{X}_H = \partial_2 H \cdot \partial_1 - \partial_1 H \cdot \partial_2$$

is the **Hamiltonian vector field** with the Hamilton function H.

Moreover, I assume that the domain U in which the considered objects (functions, vector fields, differential forms) are defined is an open contractible subset of $\mathbb{R}^2$.

The following statement (a version of the Poincaré lemma) justifies the splitting.

Lemma 1. *If the* **divergence**

$$\operatorname{div} \boldsymbol{V} = \partial_1 v_1 + \partial_2 v_2 \equiv 0,$$

then $\boldsymbol{V} = \boldsymbol{X}_H$ *is a Hamiltonian vector field.*

If $\boldsymbol{V} = g \cdot \partial_2$ *then* $\operatorname{div} \boldsymbol{V} = \partial_2 g$.

It follows that the intersection of the two summands is $C^k(\mathbb{R}_{x_1})\,\partial_2$.

Since in the Navier–Stokes system we have $\operatorname{div} \boldsymbol{V} = 0$, we can assume

$$\boldsymbol{V} = \boldsymbol{X}_H$$

for $H = H(t; x)$.

We get the equation

$$\boldsymbol{X}_{\dot{H}} - \boldsymbol{X}_{\triangle H} + (\boldsymbol{X}_H, \nabla)\,\boldsymbol{X}_H + \operatorname{grad} P - \boldsymbol{F} = 0.$$

When considering our splitting we need to split two summands in the left-hand side of the latter equation.

2. We split the force as follows:

$$\boldsymbol{F} = \boldsymbol{X}_K + L \cdot \partial_2.$$

3. Next is the gradient of the pressure.

We assume

$$\operatorname{grad} P = \boldsymbol{X}_Q + R \cdot \partial_2$$

for some functions Q and R. Of course, $\operatorname{curl} \circ \operatorname{grad} = 0$, where

$$\operatorname{curl}(a_1\partial_1 + a_2\partial_2) = \partial_1 a_2 - \partial_2 a_1$$

is the **curl** of a vector field. Hence

$$\operatorname{curl} \boldsymbol{X}_Q + \operatorname{curl}\,(R \cdot \partial_2) = 0.$$

Lemma 2. *We have*

$$\begin{aligned} \operatorname{curl} \boldsymbol{X}_Q &= -\Delta Q, \\ \operatorname{curl}\,(R \cdot \partial_2) &= \partial_1 R, \end{aligned}$$

where $\Delta = \partial_1^2 + \partial_2^2$ is the Laplacian. *(We assume that* Δ *is defined in the space of functions* f *on* U *with Dirichlet boundary conditions* $f|_{\partial U} \equiv 0$; *there it is invertible.)*

Therefore we get the condition

$$\partial_1 R = \Delta Q.$$

4. Let us apply the divergence operator to the both sides of the N–S equation for H; this corresponds to considering the Func$\cdot\partial_3$ part of the N–S equation. By Lemma 1(b) we get

$$\partial_2(R - L) = -\text{div}\left[\left(\boldsymbol{X}_H, \nabla\right)\boldsymbol{X}_H\right],$$

or

$$R = -\partial_2^{-1}\text{div}\left[\left(\boldsymbol{X}_H, \nabla\right)\boldsymbol{X}_H\right] + L,$$

where ∂_2^{-1} means the integration operator; thus R is defined non-uniquely, one can add to it a function of x_1.

Combining it with the above, we get also a formula for Q.

Lemma 3. *We have*

$$Q = -\partial_2^{-1}\Delta^{-1}\left[\partial_1\text{div}\left[\left(\boldsymbol{X}_H, \nabla\right)\boldsymbol{X}_H\right]\right] + \Delta^{-1}\partial_2 L,$$

where Δ^{-1} *is the inverse of the Laplacian in* $L^2(U)$.

We write

$$Q = \mathcal{Q}\,(H)\,.$$

5. Let us now split the nonlinear vector field, the Euler term.

Lemma 4. *We have*

$$(\boldsymbol{X}_H, \nabla)\,\boldsymbol{X}_H = \{\partial_2 H, H\} \cdot \partial_1 - \{\partial_1 H, H\} \cdot \partial_2,$$

where

$$\{f, g\} = \boldsymbol{X}_g f = \partial_1 f \cdot \partial_2 g - \partial_2 f \cdot \partial_1 g$$

is the Poisson bracket.

Recall that the non-Hamiltonian part of the splitting is proportional to ∂_2. Therefore the Hamiltonian part is determined by the ∂_1 part of this vector field, i.e., by

$$\{\partial_2 H, H\} \cdot \partial_1.$$

If this part is $\partial_2 G \cdot \partial_1$, for a function $G = G(x,t)$, then we get the equation

$$\partial_2 G = \{\partial_2 H, H\}$$

with the solution

$$G = \mathcal{G}(H) = \partial_2^{-1}\{\partial_2 H, H\}.$$

6. Summarizing the above we get the following

Theorem *The 2–dimensional Navier–Stokes system is equivalent to the following evolution equation for the function $H(t; x, y)$:*

$$\dot{H} = \Delta H - \mathcal{G}(H) - \mathcal{Q}(H) + K,$$

where $\mathcal{Q}(H) = Q(t; x)$ and *$\mathcal{G}(H) = G(t; x)$ are defined by the above equations, K is defined in the splitting of $\boldsymbol{F}$* and the right-hand side is defined modulo some additive constant $C(t; x_1)$.

Remark. The choice of the undetermined function $C(t; x_1)$ is rather delicate. It seems that it should be dictated by the geometry of the domain U and by the physics of the fluid.

Example 1. (**A flow due to an impulsively moved line boundary**) This flow takes place in the half-plane $U = \{x_2 > 0\} = \mathbb{R} \times \mathbb{R}_{>0}$. Assume that at time $t=0$ the line $\{x_2=0\}$ is suddenly jerked into motion in the x_1-direction with a constant velocity $W_0>0$. In [LK] we find the following analysis.

Due to the symmetry and 'additional preconceptions' one can assume that the flow is of the form

$$\boldsymbol{V}=W\left(t;x_2\right)\partial_1,$$

i.e., with $\operatorname{div}\boldsymbol{V}=0$. Substituting it into the N–S system we obtain the equations

$$\dot{W}=\partial_2^2W-\partial_1P,\ 0=-\partial_2P.$$

As W depends only on x_2 and t, it follows that ∂_1P depends only on t; assume additionally that $\partial_1P\equiv 0$. Then one arrives at the IVP for the heat equation:

$$\dot{W}=\partial_2^2W,\ W\left(0;x_2\right)=W_0,$$

with the solution

$$W=W_0\left\{1-\pi^{-1/2}\int_0^{2x_2/\sqrt{t}}\exp\left(-\tau^2/4\right)\mathrm{d}\tau\right\};$$

one uses the invariance with respect to the rescaling $t\longmapsto\alpha^2t$, $x_2\longmapsto\alpha x_2$ and the invariant variable $\eta=x_2/\sqrt{t}$.

Using the Hamiltonian approach we have

$$H(t;x) = \int_0^{x_2} W(t;s)\,\mathrm{d}s.$$

As grad $P = 0$, we have $Q - R = 0$. Next, $(\boldsymbol{V}, \nabla)\,\boldsymbol{V} = W\partial_1 W \cdot \partial_1 \equiv 0$; so, also $G = 0$. We are left with the heat equation $\dot{H} = \partial_2^2 H$, equivalent to the above equation for W.

Example 2. (**Poiseuille flow**) We have the domain $U = \{0 < x_2 < 1\} = \mathbb{R} \times (0,1)$. One considers the stationary (not depending on t) divergence free flow

$$\boldsymbol{V}(x) = W(x_2)\,\partial_1$$

with homogeneous boundary conditions $W(0) = W(1) = 0$. The stationary N–S equations are following:

$$0 = \partial_2^2 W - \partial_1 P,\ \ 0 = -\partial_2 P.$$

It follows that the the two summands in the right-hand side of the first equation are constant (as functions of x_2 and x_1 respectively). Thus

$$\partial_2^2 W = D = \text{const},\ \ \partial_1 P = D,$$

and hence

$$W(x_2) = \frac{1}{2} D x_2 (x_2 - 1), \; P(x_1) = D x_1.$$

From our point of view we have

$$H = D\left(\frac{1}{6}x_2^3 - \frac{1}{4}x_2^2\right), \; Q = D x_2, \; R = 0, \; G = 0.$$

Consider the N–S system in the rectangle $U = (0, \pi) \times (0, \pi)$.

Here we have our theorem with the function $C(x_1)$, which is not fixed. In fact, there are no a priori restrictions for this function and this fact is a source of an additional non-uniqueness phenomenon.

But one should made some physically sensible choice. I propose the standard averaging, i.e.,

$$Q(x) = \widetilde{Q}(x_1, x_2) - \frac{1}{\pi}\int_0^\pi \widetilde{Q}(x_1, u)\,\mathrm{d}u, \quad G(x) = \widetilde{G}(x_1, x_2) - \frac{1}{\pi}\int_0^\pi \widetilde{G}(x_1, u)\,\mathrm{d}u.$$

where $\widetilde{S}(x_1, x_2)$ and $\widetilde{G}(x_1, x_2)$ are primitives, as functions of x_2 with fixed x_1, of

$$q(x_1, x_2) = -2\partial_1\left[(\partial_1\partial_2 H)^2 - \partial_1^2 H \cdot \partial_2^2 H\right] \text{ and } g(x_1, x_2) = \partial_1\partial_2 H \cdot \partial_2 H - \partial_2^2 H \cdot \partial_1 H$$

respectively, e.g., $\partial_2 \widetilde{Q} = q$.

Example 3. (New) Take the initial vector field as $\boldsymbol{V}_0 = \boldsymbol{X}_{H_0}$ with

$$H_0 = \varepsilon \cdot \sin(m x_1) \sin(n x_2).$$

If there were no Euler and pressure terms then the solution would be $V = \boldsymbol{X}_{H(t,x)}$, where

$$H(t,x) = \mathrm{exp}\{-(m^2+n^2)t\}H_0(x).$$

With this H calculations give

$$q = -\varepsilon^2 \cdot \mathrm{e}^{-2(m^2+n^2)t} \cdot mn^2 \cdot \sin(2mx_1)$$

and

$$g = \varepsilon^2 \cdot \mathrm{e}^{-2(m^2+n^2)t} \cdot mn^2 \cdot \sin(2mx_1)\,.$$

This implies that the contribution of S and G cancel themselves and $\boldsymbol{V} = \boldsymbol{X}_H$, with H defined above, is the solution of the IVP for the N–S system with $\boldsymbol{V}_0 = \boldsymbol{X}_{H_0}$.

Sketch of the proof of the existence of solutions of the IVPs

Here I mean solutions to evolution equations for H and for $\boldsymbol{H}$.

This proof repeats the proof of the analogous problem for the **reaction–diffusion equation**

$$\dot{u} = \Delta u + u^2, \quad u(0,x) = u_0(x),$$

in $(0,\pi)$ (i.e., with $d = 1$).

From the theory of ODEs we know that the IVP

$$\dot{u} = Au + b(t), \; u(0) = u_0,$$

where A is a constant operator, has the unique solution

$$u\,(t) = \mathrm{e}^{t\cdot A} u_0 + \int_0^t \mathrm{e}^{(t-\tau)\cdot A} b(\tau) \mathrm{d}\tau.$$

We apply it to our situation with $A = \Delta$ and $b\,(t) = \phi^2\,(t;x)$, where $\phi\,(t;x) = \phi\,(t)$ is the solution. We obtain the equation

$$\phi\,(t) = \mathrm{e}^{t\cdot\Delta} u_0 + \int_0^t \mathrm{e}^{(t-\tau)\cdot\Delta} \phi^2\,(\tau)\,\mathrm{d}\tau,$$

or

$$\phi = \mathrm{e}^{t\cdot\Delta} u_0 + Q_t(\phi, \phi),$$

where $Q_t(\cdot,\cdot) = Q(\cdot,\cdot)$ is a bilinear operator valued form.

The latter equation is an equation for fixed point of the operator $\mathrm{e}^{t\cdot\Delta} + Q_t$. It is quite natural to solve it by iterations:

$$\begin{aligned} \phi_0(t) &= \phi_0 = \mathrm{e}^{t\cdot\Delta} u_0, \\ \phi_{k+1} &= \phi_0 + Q(\phi_k, \phi_k), \; k = 1, 2, \ldots. \end{aligned}$$

This sequence converges, but in some special sense.

I plan to write a separate paper devoted to this subject.

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Thank you very much for your attention!