Extension of Lipschitz maps definable in Hensel minimal structures

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Krzysztof Jan Nowak

Instytut Matematyki, Uniwersytet Jagielloński

E-mail: nowak@im.uj.edu.pl

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Some history about Lipschitz extension

The extention of Lipschitz continuous functions $f : A \to \mathbb{R}$ from a subset A of $\mathbb{R}^n$ with the same constant, say 1, goes back to McShane and Whitney. The more difficult and delicate case of functions with values in $\mathbb{R}^k$ was achieved by Kirszbraun (1934).

A purely topological, non-Archimedean version of Kirszbraun's extension theorem for spherically complete, rank one valued fields K was achieved by means of the Kuratowski–Zorn lemma by Bhaskaran (1983). And Lipschitz extension on ultrametric spaces, with an arbitrarily small magnification of the Lipschitz constant, was given by Brodskiy–Dydak–Higes–Mitra (2007).

Aschenbrenner–Fischer (2011) achieved a definable, real version of Kirszbraun's theorem for definably complete expansions of ordered fields. The only definable, non-Archimedian version of Kirszbraun's theorem was achieved by Cluckers–Martin [CM] in the p-adic, thus locally compact case; and more precisely, for functions which are semi-algebraic or subanalytic over a finite extension of the field $\mathbb{Q}_p$.

They prove this theorem and the existence of a definable 1-Lipschitz retraction for any closed definable subset A of $\mathbb{Q}_p^n$, proceeding with simultaneous induction on the dimension n of the ambient space. Their construction of definable retractions makes use of some definable Skolem functions.

Also, they posed the question (see ibid. Remark 3) whether their p-adic version of Kirszbraun's theorem and the existence of Lipschitz retractions for p-adic closed definable subsets hold in some form for other classes of valued fields; with natural examples $\mathbb{R}((t))$ and $\mathbb{C}((t))$. And they indicated that some difficulties in more general settings are the absence of definable Skolem functions in general and infiniteness of the residue field. The easier case of Lipschitz extension of definable p-adic functions on the affine line $\mathbb{Q}_p$ was treated in [K].

Generally, local Lipschitz continuity does not imply piecewise Lipschitz continuity in the absence of definable algebraic Skolem functions. This makes the problem of definable Lipschitz extension subtler yet, and is reflected in the construction of a package itself.

Main aim

Theorem

Let $f : A \to K^m$ be a 0-definable 1-Lipschitz map on a (possibly non-closed) subset $A \subset K^n$ of dimension k.

I. Suppose the value group $|K|$ has no minimal element among the elements > 1. Then, for any $\epsilon \in |K|$, $\epsilon > 1$, f extends to a 0-definable ϵ-Lipschitz map $F : K^n \to K^m$.

II. Suppose the value group $|K|$ has the minimal element ϵ among the elements > 1. Then f extends to a 0-definable $\epsilon^{\omega(k)}$-Lipschitz map $F : K^n \to K^m$, where $\omega(k) = 2^{k-2}$ if $k \geqslant 2$, and $\omega(1) = \omega(0) = 0$.

Actually, for the sake of the proof, we need the stronger version, which is uniform with respect to parameters from the sort RV.

Magnification of the Lipschitz constant

It is caused by that the initial 1-Lipschitz function f is replaced by a function g which satisfies the condition:

$$rv(g(x_1, \ldots, x_{k-1}, x_k + y_k, x_{k+1}, \ldots, x_n) - g(x)) = rv(y_k)$$

for a single, distinguished variable x_k; most often it is the last variable of the package under consideration. In other words, g is a risometry onto its image with respect to the variable x_k, the concept introduced by Halupczok.

The key property of a risometry we use is that the image of an open ball is an open ball of the same radius.

We shall prove the above extension theorem by double induction on the dimension n of the ambient space K^n and on the dimension k of the subset $A \subset K^n$. Each induction step of type (n, k), with $n \geqslant 1$ and $1 \leqslant k \leqslant n$, requires the induction hypothesis of type $(n, k-1)$ (ordinary version) and of type $(k-1, k-1)$ (uniform version).

Lipschitz extension property

Each induction step, both of the first and second kind, increases the Lipschitz constant by any factor $\epsilon > 1$ if the value group vK has no minimal element among the elements > 1, and by some power ϵ^ω if ϵ is the minimal element from among the elements > 1.

The exponent $\omega = \omega(n, k)$ depends on the type (n, k) of induction step. We have the following formulae

$$\omega(n, 0) = 0, \quad \omega(n, k) = \omega(n, k-1) + \omega(k-1, k-1) \text{ if } 1 \leqslant k \leqslant n;$$

hence $\omega(n, k) = \omega(k) = 2^{k-2}$ if $k \geqslant 2$, and $\omega(n, 0) = \omega(n, 1) = 0$.

Proposition

Consider a finite number of 0-definable subsets $A_1, \ldots, A_s$ of K^n. If each of them has the LE-property, so does their union $A_1 \cup \cdots \cup A_s$.

Cell decomposition compatible with RV-parameters

Theorem

For every 0-definable sets

$$X \subset K^n \ \ \text{and} \ \ P \subset X \times RV(K)^t,$$

there exists a finite decomposition of X into 0-definable reparametrized cells (C_k, σ_k) such that the fibers of P over each twisted box of each C_k are constant or, equivalently, the fiber of P over each $\xi \in RV(K)^t$ is a union of some twisted boxes from the cells C_k.
Furthermore, one can require that each C_k is, after some coordinate permutation, a reparametrized cell of type $(1, \ldots, 1, 0, \ldots, 0)$ with 1-Lipschitz centers

$$c_\xi = (c_{\xi,1}, \ldots, c_{\xi,n}), \ \ \xi \in \sigma(C).$$

The concept of a package with a skeleton

For the case $k > 0$, we introduce the concept of a 0-definable open cell package $C \subset K^k$ induced in a canonical way by a given parametrized open cell. C is determined by a skeleton which consists of the following sequences of centers

$$(c_{j,1}),\ j = 1, \ldots, d_1,\ \ldots,\ (c_{j,k}(x_1, \ldots, x_{k-1})),\ j = 1, \ldots, d_k,$$

and an infinitary part

$$R_{j_1,\ldots,j_k} \subset G(K)^k, \quad j_i = 1, \ldots, d_i,\ i = 1, \ldots, k,$$

which yield the presentation

$$C = \bigcup_{j_1,\ldots,j_k} C_{j_1,\ldots,j_k} \ \text{ with } \ C_{j_1,\ldots,j_k} :=$$

$$\{x \in K^k :\ (rv(x_1 - c_{j,1}), \ldots, rv(x_k - c_{j,k}(x_1, \ldots, x_{k-1}))) \in R_{j_1,\ldots,j_k}\}.$$

Moreover, each center is 1-Lipschitz on every relevant twisted box determined by the above presentation.

The basic idea of the proof

The skeleton is defined recursively to satisfy a metric condition relating the distance between centers and the size of the twisted boxes. This construction allows us to proceed with induction and Lipschitz extension.

The basic idea is first to construct a 0-definable Lipschitz extension on the graphs of the final centers of the package, and eventually, making use of the ordinary induction hypothesis, to construct the desired extension on the affine space.

To this end, we still need the following package property:

$$R_{j_1,\ldots,j_k} \subset \{\lambda \in G(K)^k : \; |\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_k|\}$$

for all $j_1, \ldots, j_k$, which can be obtained by reordering the variables by means of the uniform version of the induction hypothesis.

I should emphasize that the above mentioned constructions and the construction of a package with a skeleton itself are quite technical and require some extra effort and new ideas.

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