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Autoreport

Introduction

The main part of my scientific interest and activity concentrates on gradients.

Gradients or generalized gradients in the sense of Stein and Weiss are first order differential operators that are irreducible summands of the covariant derivative ∇. More exactly, if one starts from any linear bundle E over M, a differential manifold, and terminates together with ∇ in the bundle $T^*M \otimes E$ and if, additionally, one has a Lie group $\mathfrak{G}$ acting both on E and $T^*M \otimes E$ (and such a group is always strictly associated to the geometric structure considered on M), then one can think on splitting both the origin bundle E and the target bundle $T^*M \otimes E$ onto direct sums of $\mathfrak{G}$–irreducible invariant sub-bundles. The restriction of ∇ to any one of such sub-bundles of E composed with the projection onto any one of $E \otimes T^*M$ is just a $\mathfrak{G}$*–gradient*. We are mainly interested in SO(n)–gradients, i.e., in the case $\mathfrak{G} = \mathrm{SO}(n)$ (sometimes also in the case $\mathfrak{G} = \mathrm{GL}(n)$). The exact definition is given in the next section.

Roughly speaking gradients are the simplest bricks the covariant derivative is build of.

SO(n)–gradients were introduced in 1968 by E. Stein and G. Weiss in their famous paper *Generalization of the Cauchy-Riemann equations and representations of the rotation group* [50]. Their theory developed next into a large branch of global analysis, geometry, differential operators or the representation theory. Many natural first order linear differential operators in Riemannian geometry are either gradients or their compositions. For example, the exterior and interior derivatives d and d^*, respectively, the Cauchy-Riemann operator $\overline{\partial}$ are gradients while the classical Dirac operator on exterior forms, namely, $d+d^*$ is their sum. The composition of the operator $d+d^*$ with its adjoint leads to the Hodge Laplacian on skew symmetric (exterior) forms $\Delta = d^*d + dd^*$.

Gradients depend on the geometry of M (the group $\mathfrak{G}$) and this is obvious, but, on the other hand, they can themselves, e.g., by their spectral properties, determine, to some extent the geometry (volume, area of the boundary, scalar curvature), The nice algebraic properties of gradients make that their theory is still successfully developing.

Generally my scientific work with the gradients splits into two parts:

- Investigating of the Cauchy-Ahlfors operators - which is one of the gradients - in the context of its applications in the theory of conformal and quasiconformal deformations and transformations.

- Investigations of gradients - as the family of differential operators with interesting analytic, algebraic and geometric properties - their ellipticity, their behavior at the boundary, their geometry.

The two parts are disjoint when they are considered with respect the subject matter and they are overlapping when they are considered with respect the the date of publication.

The first part terminates with publishing of my habilitation paper [45] in 1997. The other starts in 1996 with publishing the papers [23] and [24].

The two of my papers [23] and [24] are not cited in the habilitation work [23]. Since the subject of the two papers is inseparably connected to my "after habilitation activity" I will place them (in this report) to this part of activity.

In my scientific work I have cooperated with many mathematicians: Paweł Walczak, Adam Bartoszek, Wojciech Kozłowski, Małgorzta Ciska and Anna Kimaczyńska (University of Lodz) , Bogdan Balcerzak and Jerzy Kalina (Technical University of Lodz), Bent Ørsted (Odense University and next Aarhus University, Danmark), Thomas. P. Branson (Copenhagen University, Denmark and next The University of Iowa, USA), Peter B. Gilkey (The Oregon University, USA) and Genkai Zhang (Odense University and next University of Gothenburg, Sweden).

The report consist of three parts

1. The overview of selected papers
2. The overview of selected results
3. The scientific CV

In the first one, the short summaries of selected 21 papers are given.

In the second one, the selected results are given as theorems. There are 19 of them. To make them more understandable, some theoretical material with necessary definitions and facts is add. In this part my author and co-author theorems are distinguished with the headline: Theorem (Lemma etc.) and the numeral from the references. The theorems of other authors are only distinguished with italics.

After this two parts the list od references is placed.

In the third part my scientific CV is given.

The report closes the list of my publications.

1. The overview of selected papers

1.1. Selected papers before the habilitation and the habilitation paper

The papers: [40], [41], [42] and [43].

L. Ahlfors - in his papers papers [1] [2], [3] - studied the first order linear operator S acting on a vector fields in $\mathbb{R}^n$ as follows: SX is the matrix field which - for any vector field $X = [X_1, \ldots, X_n]$ - is the matrix field with

$$(SX)_{ij} = \frac{1}{2}(\frac{\partial X_i}{\partial x^j} + \frac{\partial X_j}{\partial x^i}) - \frac{1}{n}\delta_{ij}\sum_{k=1}^{n}\frac{\partial X_k}{\partial x^k}, \tag{1}$$

The operator SX plays there an essential role in the theory of of quasiconformal deformations. The role comes from the fact that the norm of SX is a good measure of quasiconformality:

If $|SX| \leqslant k$ then X generates the one parameter family of quasiconformal transformations $(\Psi_t)_{t\in\mathbb{R}}$ such that the constant of quasiconformality of Ψ_t is bounded by exp($\frac{1}{2}k^2$).

I have noticed that S, when considered as a more general object, namely as an operator on a Riemannian manifold M, $\dim M = n$, with a Riemannian metric g and its Levi-Civita covariant derivative ∇ becomes the symmetric and trace free part of ∇ and that is has many nice geometric properties. In particular it is conformally covariant and its kernel consists of conformal vector fields. Moreover, for a given deformation (=vector field) X on M the norm of the tensor field SX appeared to be "good measure" for the constant of quasiconformality $c(X)$- an invariant known from the theory of quasiconformal deformations. I started a systematic study of S on a Riemannian manifold and its relation to the theory of quasiconformal deformations and transformations. I have noticed that S is an elliptic operator in the sense of injectivity of its symbol and that it belongs to the class of so called *Stein-Weiss operators*, an important class of first order differential operators related to the action of special orthogonal group $\mathrm{SO}(n)$. The results of the investigations were published in my papers [40], [41]. The results from these papers were used in investigations of the rank of quasiconformality for pseudo-conformal deformations of real hypersurfaces in the complex space $\mathbb{C}^{n+1}$. The results were published in "Mathematica Scndinavica" [42].

A further investigations of the operator S - called today the *Ahlfors* or the *Cauchy-Ahlfors* operator - the adjoint S^* operator and the composition S^*S being a second order strictly elliptic operator revealed several interesting geometric properties. An application of the known Weitzenböck formula resulted in getting an important formula relating S^*S to the Hodge-Laplacian Δ and the Ricci curvature on of M [43].

The paper [44].

The transfer of the Cauchy-Ahlfors operator S from $\mathbb{R}^n$ and onto a Riemannian manifold opened a new fields for its study. First of all the spectral properties can be investigated. Indeed, in the compact case the strongly elliptic operator S^*S has a discrete spectrum. The results from [43] relating S^*S to the Hodge Laplacian the Ricci curvature enable now to determine the spectrum in many particular cases and getting important estimations on the norm of S in the dependence on the curvature. Using the fact that the norm of SX is the measure of the rank of quasiconformality for a deformation X and using the spectral properties of the selfadjoint strongly elliptic operator S^*S I established several estimations for the constant of quasiconformality for several classes of deformations, I also obtained the lower bounds for constant of quasiconformality of an arbitrary deformation on M in the dependence on the Ricci curvature. The results were published in "manuscripta mathmatica" [44].

The paper [33].

Together With B. Ørsted we investigated the asymptotic expansion of the heat kernel for the Ahlfors Laplacian S^*S on a closed Riemannian manifold and obtained its asymptotic expansion. We also determined the spectrum in the case of the euclidean sphere [33].

The papers [14], [15].

With T. P. Branson i P. B. Gilkey and next also with B. Ørsted we determined several initial coefficients of the asymptotic expansion of the heat kernel for second order linear operators with the symbol of non-metric type and under different types of boundary conditions [14] [15].

The paper [34].

With B. Ørsted we investigated the boundary value problem for S^*S for several physically motivated boundary conditions on an arbitrary compact Riemannian manifold with the boundary. We proved the ellipticity in the sense of Gilkey-Smith [19] of some natural boundary conditions. The considered conditions were similar to the conditions studied by H. Weyl in 1915 [51] where he studied the theory of the elastic body in $\mathbb{R}^3$. For each of our boundary condition we obtained the asymptotic distribution of the eigenvalues for S^* what was a far generalization of the deep result by H. Weyl from 1915.

The habilitation paper [45].

My habilitation dissertation [45] is continuation of my paper from ”manuscripta mathematica” [45]. Several results on relations between the constant of quasiconformality of a deformation and the geometry of manifold (curvature), in the case of a compact Riemannian manifold with the boundary was derived there. In this case the constants of quasiconformality depend not only on the Ricci curvature but also on the boundary shape (the second fundamental form) and on the boundary conditions imposed on the deformation.

1.2. Selected papers after the habilitation paper

The papers [23], [24].

Together with J. Kalina and P. Walczak we undertook the open problem of detection and classification of the elliptic gradients. For such a gradient G the second order operator G^*G is strongly elliptic operator usually encoding some information on the geometry of manifold. We succeeded by formulating and proving in the language of the Young diagrams the rule of detecting the elliptic gradients and showing that there is only one such gradient in each decomposition [23].

The result was essentially strengthen in an additional cooperation with B. Ørsted and G. Zhang [24]. The reformulation of the problem in the language of representations gave it a new meaning and enabled the extension the result onto much larger class of bundles and operators.

It is worth to add that the results obtained both in [23] and [24] not only assure the existence and the uniqueness of the elliptic gradient in the decomposition (of the covariant derivative into irreducible pieces) but also give simple rules for pointing it.

Inspired by our results, T. Branson gave a year later a full classification of the elliptic gradients and its combinations in his paper [12].

These three papers completely solved the problem of ellipticity.

The unpublished manusctript.

Together with T. Branson we cooperated also on constructing possibly universal and complete systems of elliptic boundary conditions for any elliptic gradient on a Riemannian manifold with the boundary. In particular during may stay in the University of Iowa (as *visiting associate professor*) in 2003/2004. The cooperation was continued also after my leaving. A sudden and premature passing away of T. Branson in 2006 terminated the cooperation. Some of our ideas on the elliptic boundary conditions were written in an uncompleted manuscript. It contained in particular the construction of system od natural boundary conditions but practically no theorems. Our hypothesis was: *for any given elliptic gradient all the conditions are elliptic*. The construction is universal and works well for any $SO(n)$–gradient not necessarily elliptic. Let us call this system of natural boundary conditions by (SNBC). The key role in the construction is played by the Stokes type integral formula from the paper [34] by B. Ørsted and myself (see the formula (21) below). The method of construction was next described in the introduction to my joint paper with W. Kozłowski [30] in 2008 with a note on the Branson's co-authorship. In the paper the four natural boundary conditions from (SNBC) are investigated for the weighted form Laplacian.

The paper [30].

The boundary value problem for the system of natural boundary conditions (SNBC) was solved for the weighted Laplacian

$$\Delta_{ab} = a\delta d + bd\delta \qquad a, b > 0$$

and the bundle of skew-symmetric tensors of any degree with polynomial coefficients in the euclidean ball in $\mathbb{R}^n$ in a cooperation with W. Kozłowski. There are four natural conditions independently of the degree of form: Dirichlet, absolute, relative and Neumann $\{\mathcal{D}, \mathcal{A}, \mathcal{R}, \mathcal{N}\}$. The study of the Dirichlet condition in this case was the subject of the PhD thesis of the first author [29]. Here, the conditions $\{\mathcal{A}, \mathcal{R}, \mathcal{N}\}$ are investigated. Excluding some exceptional cases the theorems on the existence and uniqueness of the solutions are proved. In the exceptional cases necessary and sufficient conditions for the the existence of solutions are formulated and the problem of uniqueness is discussed. The most interesting is the forth boundary condition $\{\mathcal{N}\}$. It is elliptic. It was not studied before in the literature. For the existence of solutions some additional assumptions have to be imposed in this case. The assumptions and the uniqueness of solutions is also discussed in this case. All the proofs on the existence for all the investigated boundary conditions are constructive. So, for a given boundary data, the solution can be derived explicitly in each case.

The paper [27].

The system of natural boundary conditions for the operator

$$\mathrm{div}\,\mathrm{grad}$$

in the bundle of symmetric tensors (forms) of any degree on a Riemannian manifold was investigated in a cooperation with A. Kimczyńska. The operator was introduced and investigated earlier in her PhD dissertation [26]. For the symmetric tensor of degree k the system (SNBC) consist of 2^{k+1} boundary conditions. This is in contrast to the skew-symmetric case, where independently of the degree there always four conditions. Our paper contains the proof of the ellipticity of all the 2^{k+1} conditions. In a contrast to the until now known proofs of the ellipticity for other bundles where the proofs are constructed to each the boundary condition separately we were able be to prove the ellipticity of all the boundary conditions simultaneously.

The papers [6], [7], [8].

Lie algebroids seem to be a good environment (object) for investigating gradients. Both algebroids and gradients have an analytic origin and both can be distinguished by their algebraic properties. There is one more reason: the notion of Lie algebroid is very capacious and includes e.g.: Lie algebras, integrable distributions, in particular foliations, cotangent bundles of Poisson manifolds and so on.

In the first papers of the series written together with B. Balcerzak, and J. Kalina, a version of the Weitzenbök formula - the fundamental tool for investigating the geometry of differential operators is proved [6].

In the two other papers together with B. Balcerzak [7] and [8] detailed constructions of all the possible gradients for the bundle of skew-symmetric forms and for symmetric forms are given. The construction of the Dirac operator on the Lie algebroid is also given explicitly. The basic algebraic and geometric properties of the constructed operators are proved.

The papers [9], [10].

Differential operators, so in particular the gradients, restrict rather badly to submanifolds and then to foliations. In our papers the difficulty was overcome by restricting considerations to the class of so called $\mathrm{SL}(q)$ foliations. This class was introduced by Ph. Tondeur in his book [49]. Roughly speaking a foliation of the class is given locally by a submersion that is transversally volume preserving. Usually the authors make much stronger assumption in a similar situation, namely they assume that the considered foliation is Riemannian. In a rough language this means that the foliation is given locally by a submersion that is transversally isometric. In a comparison to the last assumption the our one is much, much weaker. Moreover, it looks like that it is as the weakest of all possible ones. Notice that for any $q = 1, 2 \ldots$ the codimension of SL(q) in GL(q) is one! So, under this weak assumption on a foliation we get a satisfactory and coherent theory of gradients without losing their most important geometric properties. The assertions of our theorems on the existence and the shape of the formally adjoint operators are not weaker than that of other authors working with similar operators under much stronger assumption that the foliation is Riemannian. And this everything thanks to the theorem on the existence of a special coordinate system for the $SL(Q)$ foliation. The existence of this special coordinate system on the $\mathrm{SL}(q)$–foliations was proved together with A. Bartoszek and J. Kalina [9]. Working in this system we were able to get many geometrically important relations and facts. In particular the foliated Weitzenböck type formula on $\mathrm{SL}(q)$–foliated manifolds. In the other paper [10] published a year later in the Journal of Geometry and Physics the foliated gradients and the Dirac type operator were successfully defined and investigated.

The paper [16].

A conformal map $\Phi = (\varphi, \psi) : U \to V$ between open subsets U, V of Euclidean plane $\mathbb{R}^2$ can be characterized by the geometric conditions

$$|\nabla\varphi|^2 = |\nabla\psi|^2 \neq 0, \quad < \nabla\varphi, \nabla\psi >= 0 \tag{2}$$

where $< \cdot, \cdot >$ denotes the standard inner product. Since a conformal map on a plane is homomorphic or antiholomorphic, it follows that φ and ψ are harmonic. Functions φ and ψ satisfying (2) are called conjugate. The level sets of such functions compose a pair of mutually orthogonal families of curves. Their conformal moduli (being in fact important conformal invariants) are then inverse each to the other, i.e. their product is equal to 1. This express mathematically the content of the Dirichlet-Thomson principle known from physics. By abstracting the above geometric meaning of the conformality, we can formulate the idea in a very general situation. For a pair of real numbers $p, q > 1$

we define and investigate pairs of (p, q)–conjugate submersions of a Riemannian manifold. We prove that conjugate submersions of the plane are p– and q–harmonic maps, respectively if and only if $\frac{1}{p} + \frac{1}{q} = 1$. With this assumption on p and q we prove also that in the case on an arbitrary Riemannian manifold, the product of moduli of foliations defined by (p, q)–conjugate submersions is equal to 1. Moreover we prove that under a weaker assumption that the foliations are defined by such submersions merely locally, the product is less or equal to 1.

2. Overview of selected results

2.1. Gradients: the notion and examples

The main part of my scientific activity is placed in the theory of gradients. To review may achievements in this range in a more strict form (theorems) I will introduce the necessary theoretic material. Next I will formulate the three main problems of the theory. Finally I will try to describe my attempts for solving them discussing each problem separately.

Let M be a finite dimensional manifold of dimension n and let E be a vector bundle over M.

Assume that ∇ is a covariant derivative in E, i.e., assume $\mathfrak{G}$ is a Lie group acting both on T^*M and E (such a group is always strictly associated to the geometric structure considered on M).

The covariant derivative ∇ is, by definition, a first order linear differential operator

$$\nabla : E \to E \otimes T^*M. \tag{3}$$

Our notation convention here is that a bundle itself and the spaces of sections of the bundle is denoted by the same letter. For example, the symbols E or $E \otimes T^*M$ denote the bundles themselves and the spaces of their sections: $C^\infty(E)$ or $C^\infty(E \otimes T^*M)$, respectively. Of course ∇ above is the operator between the spaces of sections. We hope that the proper understanding of the symbol will easily come from the context in each case.

Split both the origin bundle E:

$$E = V_1 \oplus \cdots \oplus V_\mu \oplus \cdots \oplus V_r \tag{4}$$

and the target bundle $F = E \otimes T^*M$:

$$F = W_1 \oplus \cdots \oplus W_\nu \oplus \cdots \oplus W_s \tag{5}$$

into a direct sum of $\mathfrak{G}$–irreducible invariant subbundles of E and F, respectively. Notice that (4) and (5) define uniquely analogous splittings for the spaces of sections.

The restriction of ∇ to any one of such subbundles of E composed with the projection onto any one of F is just a $\mathfrak{G}$*–gradient*, shortly, *gradient*.

The described situation can be illustrated by the following diagram:

$$\begin{array}{ccccccc} V_1 & & & & & & W_1 \\ \oplus & & & & & & \oplus \\ \vdots & & & & & & \vdots \\ \oplus & \searrow & & & & \nearrow & \oplus \\ V_\mu & \rightarrow & E & \xrightarrow{\nabla} & F & \rightarrow & W_\nu \\ \oplus & \nearrow & & & & \searrow & \oplus \\ \vdots & & & & & & \vdots \\ \oplus & & & & & & \oplus \\ V_r & & & & & & W_s \end{array}$$

So - for any μ, ν - the first order differential operator

$$\nabla^{\mu\nu} = P_{\mu\nu} = \pi_\nu \circ \nabla \circ j_\mu : V_\mu \longrightarrow W_\nu$$

is a $\mathfrak{G}$–gradient.

The arrows terminating at E (in the left part of the diagram) represent the natural injections defined by the splitting (4). The arrows starting from F (in the right part of the diagram) - the natural projections defined by the splitting (5). To get a gradient choose one of the injections, compose it with ∇ and next with one of the projections.

In this paper we will mainly be interested in $\mathrm{GL}(n)$– and $\mathrm{SO}(n)$–gradients, i.e., in the case $\mathfrak{G} = \mathrm{GL}(n)$ or $\mathfrak{G} = \mathrm{SO}(n)$. In the other case we assume that the manifold M is oriented and equipped with a Riemannian structure represented by a scalar product $g =< \cdot, \cdot >$.

2.2. Gradients in the tensor bundles

Consider now the particular case: the origin bundle E is a tensor bundle over M. For any $k = 0, 1, 2 \ldots$, ∇ can be treated as the operator

$$\nabla : \Gamma(\bigotimes^k T^*M) \to \Gamma(\bigotimes^{k+1} T^*M).$$

By limiting considerations to this case we can derive many explicit formulas for gradients for many irreducible subbundles of $\bigotimes^k T^*M$. In particular in the bundles of skewsymmetric- or trace free symmetric tensors.

The fibers of TM are Euclidean spaces, $\mathrm{SO}(n)$ acts on them in a natural way. Obviously, the action can be extended naturally to $\bigotimes^k T^*M$.

Decompose the space $T^*M^k = \bigotimes^k T^*M$ into a direct sum of irreducible invariant subspaces:

$$T^*M^k = \bigoplus_\mu V_\mu.$$

For every μ, denote by $j_\mu : V_\mu \to T^*M^k$ the natural injection defined by the splitting.

Next, take any μ and split the bundle $V_\mu \otimes T^*M$ into a direct sum of invariant irreducible subbundles

$$V_\mu \otimes T^*M = \bigoplus_\nu W_\nu.$$

For every ν, denote by $\pi_\nu : V_\mu \otimes T^*M \to V_\nu$ the natural projection defined by the splitting.

If the multiplicities are one – and it is almost always the case in our considerations – this decomposition is unique.

A detailed information on decomposition into irreducibles of any representation (action) of $\mathrm{SO}(n)$ in a tensor bundle may be found e.g. in [52].

For any μ, ν the first order differential operator

$$\nabla^{\mu\nu} = P_{=}\pi_\nu \circ \nabla \circ j_\mu : V_\mu \longrightarrow W_\nu$$

is just a gradient.

Without loss of generality we can always confine considerations to the case when the origin bundle is irreducible:

The splitting receives then a simpler form, namely:

$$\nabla = G_1 + \cdots + G_\nu + \cdots + G_r. \tag{6}$$

From now on we will always assume that the origin bundle is irreducible.

The simplest example is the case $k = 1$. The origin bundle $\Lambda^1 = T^*M$ is irreducible ($\mathrm{SO}(n)$ acts on the unit sphere in T^*M transitively) but the target bundle $T^*M \otimes \Lambda^1 = T^*M \otimes T^*M$ splits into three $\mathrm{SO}(n)$–irreducible invariant subbundles:

$$T^*M \otimes \Lambda^1 = \bigwedge^2 \oplus \mathsf{S}^2_0 \oplus \mathsf{S}^2_{\mathrm{tr}}, \tag{7}$$

where

$\bigwedge^2$ is the subbundle of skew-symmetric tensors,

S^2_0 is the subbundle of symmetric and trace-free tensors,

$\mathsf{S}^2_{\mathrm{tr}}$ is the subbundle of pure traces, i.e. tensors of the form cg, $c \in \mathbb{R}$.

The three projections π_1,π_2, π_3 define the following three gradients

$$G_1 = \pi_1 \nabla = \frac{1}{2} d, \qquad G_2 = \pi_2 \nabla = S, \qquad G_3 = \pi_3 \nabla = -\frac{1}{n} g\ d^*,$$

so

$$\nabla = \frac{1}{2} d + S - \frac{1}{n} g\ d^*, \tag{8}$$

where d and d^* are the operators of exterior derivative and coderivative, respectively.

2.3. The main problems of the theory of gradients

One of the interesting fact on $\mathrm{SO}(n)$–gradients is that they can be characterized by their *conformal covariance.*

The following fundamental fact was proved by Fegan [17].

Each $\mathrm{SO}(n)$*–gradient* G *is conformally covariant, in the sense that there are constants* c *and* c^* *with*

$$G = \Omega^{-(c+1)} \underline{G} \Omega^c, \qquad G^* = \Omega^{-(c^*+1)} \underline{G}^* \Omega^{c^*},$$

whenever we have two conformally equivalent metrics, i.e. metrics g *and* $\underline{g}$ *related by* $g = \Omega^2 \underline{g}$ *for some positive smooth function* Ω *on* M.

Conversely, any conformally invariant operator from an $\mathrm{SO}(n)$*–irreducible bundle is a composition of a gradient and a bundle map.*

The Fegan's theorem established a new perspective in the theory of gradients. It gave a complex unified view on the whole family of gradients and brought a hope that other interesting features can be selected and characterized.

Gradients are differential operators. Some of them are elliptic in the sense of injectivity of their symbol. Some of them are not. Can the ellipticity be the property that can be recognized and detected? Can the ellipticity be saved at the boundary if ∂M is nonempty? What boundary conditions should be imposed then?

If we start with a covariant derivative from an irreducible bundle (which, by the way, is an elliptic operator) then the splitting (6) is composed in general of both elliptic and not elliptic summands (gradients).

A natural question is: *which* and *how many* of summands are elliptic. And further, what sums or linear combinations are elliptic. Let us call this an the related questions *the problem of ellipticity.*

For example, the operator S from the decomposition (8) is elliptic. In fact, it is the only elliptic gradient in the decomposition. The two other gradients d and δ are not elliptic. But (attention!) their sum $d + \delta$ – the Dirac type operator – is.

Gradients depend on the geometry of M (the group $\mathfrak{G}$) and this geometry is very often encoded in them. The kernels of gradients have usually explicit geometric properties or interpretation. For example the kernel of the Cauchy-Ahlfors operator S consists of conformal deformations. On the other hand the gradients, can for example, by their spectral properties, determine – to some extent – the geometry (volume, area of the boundary, scalar curvature, etc.). The nice analytic geometric and algebraic properties of gradients made that their theory can also be successfully developed in other categories of objects, e.g. on Lie algebroids.

Generally the three following problems of the theory of gradients may be distinguished:

- **the problem of ellipticity**
- **the problem of constructing natural elliptic boundary conditions for the elliptic gradients**
- **the problem of the geometry of gradients (the direct and the inverse problem)**

All the three mentioned problems (domains) are, in my opinion, equally interesting and the author and his colleagues and cooperators have their small contribution in solving each of them. Our nearest aim is to review this contribution. But first of let us introduce the necessary notions and some general facts of the theory.

2.4. The ellipticity

Elliptic operators has many nice analytic and geometric properties especially from the point of view of existence or uniqueness of solutions to differential equations involving such operators.

Now, similarly to [32], introduce the notions of symbol and of ellipticity.

Let E and F be two vector bundles over M and $P : E \to F$ be a linear differential operator of order m.

Let $p \in M$ and $\xi \in T_p^*M$. Let $\mathfrak{m}_p$ denote the ring of germs of smooth functions vanishing at p. Let $f \in \mathfrak{m}_p$ be a smooth function defining ξ, i.e. $\xi = df(p)$. Let $e \in E_p$ and and s be such a section of E that $s(p) = e$.

The *symbol* of P at p is the map $\sigma : E_p \times T_p^* \to F_p$ defined by

$$\sigma(e, \xi) = P(f^m s)(p). \tag{9}$$

One can prove that the definition is correct, i.e. the right hand side of (9) is independent of the choice of f and s. We will also write $\sigma(p, \xi)$ instead of $\sigma(e, \xi)$ to stress the dependence on p and ξ.

A linear differential operator P is called *elliptic at p* if the map

$$E_p \ni e \mapsto \sigma(e, \xi) \in F_p$$

is injective for every $\xi \in T_p^*$, $\xi \neq 0$.

We say that P is *elliptic* if it is elliptic at every p.

Recall that ∇ is an elliptic operator in the above sense.

Now the question arises which gradients in the splitting (6) are elliptic.

Notice that if G_ν is elliptic then the second order differential operator

$$G_\nu^* G_\nu,$$

where G_ν^* denotes the operator formally adjoint to G_ν, is strongly elliptic.

The problem of ellipticity was completely solved within the three following years: 1995, 1996 and 1997 in the three following papers: [23], [24] and [12].

The first answer to the question on the ellipticity of particular gradients was given by J. Kalina, A. Pierzchalski, P. Walczak in [23]. The paper was sent for publication in 1995.

It was proved there that in the case of $\mathfrak{G} = GL(n)$ and ∇ starting from $\mathfrak{G}$–irreducible bundle there is exactly one elliptic gradient in the splitting (6). To detect the elliptic gradient we used the Young diagram method. Let us describe it shortly.

Let W be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) of dimension n. Fix $k \in \mathbb{N}$ and take a sequence of integers $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\alpha_1 \geqslant \cdots \geqslant \alpha_r \geqslant 1$, $\alpha_1 + \cdots + \alpha_r = k$. Such an α is called a *Young scheme of length k*. In some references a Young scheme is called a *decomposition*. It can be represented by a figure consisting of r rows of squares and such that the number of the squares in the j-th row is α_j.

A Young scheme can be filled with numbers $1, \ldots, k$ distributed in any order. A scheme filled with numbers is called a *Young diagram*. Without loss of generality, we can assume that the numbers grow, both, in rows and in columns.

Take a Young diagram α and denote by H_α and V_α the subgroup of the symmetric groups S_k consisting of all permutations preserving rows and columns, respectively. The diagram α determines the linear operator (called the *Young symmetrizer*) $P_\alpha : W^k \to W^k$, $W^k = \bigotimes^k W$, being the projection of W^k onto an invariant an invariant subspace of W^k for the standard representation of $GL(n)$ in W^k. This representation is irreducible on W_α. Moreover,

$$W^k = \bigoplus_\alpha W_\alpha.$$

Repeat the same construction to the space $W^{k+1} = W^k \otimes W$ to get an analogous splitting into irreducible invariant subspaces:

$$W^{k+1} = \bigoplus_{\beta} W_\beta.$$

The symbol of ∇ is just "tensoring by a covector". More exactly the action of the symbol abstracts in this case as follows.

Take arbitrary $v \in W$ and consider a linear mapping $\otimes_w : W^k \to W^{k+1}$ defined by

$$\otimes_w(w_1 \otimes \cdots \otimes w_k) = w_1 \otimes \cdots \otimes w_k \otimes w$$

.

Theorem 1. **([23])** *For any $v \neq 0$ the mapping*

$$P_\beta \circ \otimes_w|_{W_\alpha} : W_\alpha \to W_\beta$$

is injective if and only if β is the distinguished extension of α.

Recall that the Young diagram β of the length $k+1$ is the distinguished extension of α if is obtained from α by an extension by a single square. The extended diagram should have $k+1$ in the added square, while the ordering in the remaining part of the diagram is the same as in α, i.e. if

$$s = r, \quad \beta_1 = \alpha_1 + 1 \quad \beta_2 = \alpha_2, \ldots \beta_s = \alpha_s.$$

In other words, looking at the Young diagrams, the extension of α to β is distinguished when the added square is situated at the end of the first row.

Notice that Theorem 1 applied to any irreducible tensor bundle give a direct rule for determining the elliptic gradient. Notice also the striking simplicity of the rule. Let us state that as the following

Corollary 2. **([23])** *The operator (gradient) $G_{\alpha\beta} = P_\beta \circ \nabla|_{W_\alpha}$ is elliptic if and only if β is the distinguished extension of α.*

Analogous results were also obtained in [23] for some SO(n)–gradients. In particular, for such gradients in the bundle of skew-symmetric tensors and in the bundle of symmetric tensors.

A year later a more general fact was proved for the case of compact semisimple group $\mathfrak{G}$ by J. Kalina, A. Pierzchalski. B. Ørsted, P. Walczak, G. Zhang in [24].

Let $\mathfrak{G}$ be a compact semisimple Lie group and U and V two irreducible finite-dimensional unitary representations with highest weights μ and ν respectively. The tensor product $U \otimes V$ contains the unique invariant subspace W on which $\mathfrak{G}$ acts by the irreducible representation of highest weight $\mu + \nu$. The following observation was proved there.

Lemma 3. **([24])** *Let P be the orthogonal projection on an $\mathfrak{G}$–irreducible invariant subspace of the tensor product $U \otimes V$. Then the implication:*

$$u \otimes v \neq 0, \quad u \in U, v \in V \quad \Rightarrow \quad P(u \otimes v) \neq 0 \tag{10}$$

holds if and only if P is the projection to W.

This observation leads directly to an ellipticity criterion for $\mathfrak{G}$–gradients.

Theorem 4. **([24])** *Let $\mathfrak{G} \subset \mathfrak{G}_{\mathbb{C}}$ be a connected compact semi-simple group, U and V, complex representations of $\mathfrak{G}$ with the highest weights λ and μ, respectively. Then if*

$$U \otimes V = \bigoplus_i W_i, \tag{11}$$

is the irreducible decomposition of $U \otimes V$ and W_1 is the Cartan component i.e. corresponding to the highest weight $\lambda + \mu$, the projection P_1 from $U \otimes V$ onto W_1 is the only one among the P_i that satisfies the ellipticity condition (10).

There is now an immediate conclusion from proven theorems about the ellipticity of respective gradients.

Note, that similarly to the results of the paper [23], also here, from proven theorems 3 and 4, an existence of one and only one elliptic gradient in decomposition of covariant derivative. These theorems namely indicate the gradient.

In the discussed paper there is one more theorem, that I would like to mention. It gives the number of irreducible representations of unitary group $U(n)$ in the tensor product of k copies of $\mathbb{C}^n$, i.e. in the space $\otimes^k \mathbb{C}^n$. Let $d(k) = d(k, n)$ be this number.

Theorem 5. **([24])** *We have that, for $k \leqslant n$,*

$$d(k) = k! \times \textit{ coefficient of } t^k \textit{ in } \exp(t + \frac{1}{2}t^2)$$

and

$$d(k) - \textit{number of elements } \alpha \in S_k \textit{ such that } \alpha^2 = 1.$$

Moreover, we have the following recursion formula

$$d(k+1) = kd(k-1) + d(k).$$

Next, again a year later, the solution to the problem of ellipticity was completed by T. Branson with the investigating also linear combinations of gradients.

In his beautiful paper [12] he proved among other that for $\mathbb{G} = \mathrm{SO}(n)$ and the corresponding splitting of type (6) the following fact.

There are sets $B_1, \ldots, B_p \subset \{1, ..., r\}$, each of cardinality 1 or 2, such that

$$\sum_{\nu \in A} G_\nu^* G_\nu$$

is elliptic if and only if $B_u \subset A$ for some u.

Furthermore, excluding some exceptional cases the sets B_u partition $\{1, \ldots, r\}$, i.e. B_u are pairwise disjoint and $\{1, \ldots, r\}$ is the sum of all B_u.

In that way, the problem of ellipticity was solved completely.

A Lie algebroid over a manifold M is a vector bundle A over M with a homomorphism of vector bundles $\varrho_A : A \to TM$ called an *anchor*, and a real Lie algebra structure $(\Gamma(A), [\![\cdot, \cdot]\!])$ such that $[\![a, fb]\!] = f[\![a, b]\!] + \varrho_A(a)(f) \cdot b$ for all $a, b \in \Gamma(A)$, $f \in C^\infty(M)$.

Any smooth manifold M defines a Lie algebroid, where $A = TM$ with the identity anchor and the natural Lie algebra of vector fields on M. Other examples of Lie algebroids are: Lie algebras, integrable distributions, in particular foliations, cotangent bundles of Poisson manifolds, Lie algebroids of principal bundles.

Assume that A is an oriented bundle and that it is equipped with a Riemannian metric g. The metric can be transmitted to A^* and then to any tensor product of any number of copies of A^*. The extended metric will be denoted by the same letter g. Assume that rank of A is n. Each fiber is then an Euclidean space of dimension n. The group $SO(n)$ acts then on A^* and all the tensor bundles. In particular, on the bundles of skew symmetric and symmetric tensors.

Two important cases: the skew-symmetric forms and the trace-free symmetric tensors taken as the origin bundle are investigated in details. In both of them the covariant derivative splits exactly into three pieces. One of our aim is getting a possibly full analogy and harmony in description of this two quite antipodal cases.

Define the *antisymmetric-trace* operator as the usual metric trace with respect to the first two arguments:

$$\operatorname{tr}^a : A^* \otimes \bigwedge^k A^* \longrightarrow \bigwedge^{k-1} A^*$$

and the *antisymmetric-cotrace* operator

$$\operatorname{cotr}^a : \bigwedge^{k-1} A^* \longrightarrow A^* \otimes \bigwedge^k A^*$$

defined as the operator conjugate to $k \cdot \operatorname{tr}^a$ in the following sense $\operatorname{cotr}^a = k \cdot (\operatorname{tr}^a)^*$ or more exactly:

$$\langle \operatorname{cotr}^a(\eta), \xi \rangle_g = \langle \eta, k \cdot \operatorname{tr}^a \xi \rangle_g \tag{12}$$

for $\eta \in \bigwedge^{k-1} A^*$, $\xi \in A^* \otimes \bigwedge^k A^*$.

Define three linear mappings

$$\pi_1^a, \pi_2^a, \pi_3^a : A^* \otimes \bigwedge^k A^* \longrightarrow A^* \otimes \bigwedge^k A^*$$

by

$$\pi_1^a = \operatorname{Alt}, \qquad \pi_2^a = \operatorname{id} - \pi_1^a - \pi_3^a, \qquad \pi_3^a = \frac{1}{n-k+1} \operatorname{cotr}^a \circ \operatorname{tr}^a.$$

where Alt is *antisymmetrizer* acting on any $k+1$–tensor ϑ by

$$(\operatorname{Alt} \vartheta)(a_1, \ldots, a_{k+1}) = \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} \operatorname{sign} \sigma \; \vartheta\left(a_{\sigma(1)}, \ldots, a_{\sigma(k+1)}\right).$$

and the three differential operators

$$P_j^a = \pi_j^a \circ \nabla^a : \bigwedge^k A^{*k} \longrightarrow A^* \otimes \bigwedge^k A^*, \qquad j \in \{1, 2, 3\}.$$

Theorem 6. **([7])** *$\pi_1^a, \pi_2^a, \pi_3^a$ are projections and $A^* \otimes \bigwedge^k A^*$ splits onto the direct (in fact, orthogonal) sum of $SO(n)$–invariant subspaces:*

$$A^* \otimes \bigwedge^k A^* = \operatorname{Im} \pi_1^a \oplus \operatorname{Im} \pi_2^a \oplus \operatorname{Im} \pi_3^a. \tag{13}$$

The covariant derivative ∇^a acting in the bundle $\bigwedge^k A^*$ splits onto three operators

$$\nabla^a = P_1^a + P_2^a + P_3^a \tag{14}$$

of form

$$P_1^a = \tfrac{1}{k+1} d^a, \quad P_2^a = \nabla - \tfrac{1}{k+1} d^a + \tfrac{1}{n-k+1} \operatorname{cotr}^a \circ d^{a*}, \quad P_3^a = \tfrac{-1}{n-k+1} \operatorname{cotr}^a \circ d^{a*}. \tag{15}$$

where d^a and d^{a*} are operators of exterior derivative and coderivative, respectively, in the bundle of skew-symmetric forms.

Remark 2.1. For $k \neq \frac{n}{4} \neq k+1$ the orthogonal subspaces in the splitting (13) are irreducible. For $n = 4k$ or $n = 4(k+1)$ the origin bundle $\bigwedge^{\frac{n}{4}} A^*$ splits $\bigwedge_+^{\frac{n}{4}} A^* \oplus \bigwedge_-^{\frac{n}{4}} A^*$ where $+/-$ denotes the subbundles of $\bigwedge^{\frac{n}{4}} A^*$ being the eigenspaces of the Hodge star operator, respectively.

Notice that $\operatorname{Im} \pi_1^a = \bigwedge^{k+1}$. Elements of $\operatorname{Im} \pi_1^a \oplus \operatorname{Im} \pi_2^a$ are trace-free tensors, i.e. $\operatorname{tr}^a \eta = 0$ for $\eta \in \operatorname{Im} \pi_1^a \oplus \operatorname{Im} \pi_2^a$, so elements of $\operatorname{Im} \pi_3^a$ may be called *pure traces.*

Remark 2.2. To get gradients in (14) for the exceptional cases, compose P_1^a with the projections onto $\bigwedge_+^{\frac{n}{4}} A^*$ and $\bigwedge_-^{\frac{n}{4}} A^*$ for $n = 4k$, or restrict the origin bundle to one of $\bigwedge_+^{\frac{n}{4}} A^*$, $\bigwedge_-^{\frac{n}{4}} A^*$ for $n = 4\,(k+1)$.

Remark 2.3. *Notice also that P_2^a is the only elliptic operator of the three ones in the sense of injectivity of its symbol.*

Consider now the case of symmetric forms.

Define the operator *symmetric derivative* $d^s : \mathsf{S}^k A^* \longrightarrow \mathsf{S}^{k+1} A^*$ as the symmetrization of the covariant derivative ∇^s acting on the bundle $\mathsf{S}^k A^*$ of symmetric tensors (forms)

$$d^s = (k+1) \cdot (\operatorname{Sym} \circ \nabla^s) \quad \text{on} \quad \mathsf{S}^k A^* \tag{16}$$

where Sym is the *symmetrizer* acting on any $k+1$–tensor ϑ by

$$(\operatorname{Sym} \vartheta)\,(a_1, \ldots, a_{k+1}) = \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} \vartheta\left(a_{\sigma(1)}, \ldots, a_{\sigma(k+1)}\right).$$

By the *symmetric coderivative* d^{s*} we mean the restriction of the coderivative to the space of symmetric tensors:

$$d^{s*} = \nabla^*|_{\mathsf{S}^k A^*} : \mathsf{S}^k A^* \longrightarrow \mathsf{S}^{k-1} A^*. \tag{17}$$

Define two operators. The *symmetric-trace*

$$\operatorname{tr}^s : A^* \otimes \mathsf{S}^k A^* \longrightarrow \mathsf{S}^{k-1} A^*$$

as the restriction of the metric trace with respect to the first two arguments to $A^* \otimes \mathsf{S}^k A^*$ and the *symmetric-cotrace*

$$\operatorname{cotr}^s : \mathsf{S}^{k-1} A^* \longrightarrow A^* \otimes \mathsf{S}^k A^*$$

defined as the operator conjugate to $k \cdot \operatorname{tr}^s$ in the following sense $\operatorname{cotr}^s = k \cdot (\operatorname{tr}^s)^*$ or more exactly:

$$\langle \operatorname{cotr}^s(\eta), \zeta \rangle_g = \langle \omega, k \cdot \operatorname{tr}^s \zeta \rangle_g \tag{18}$$

for $\eta \in \mathsf{S}^{k-1} A^*$, $\zeta \in A^* \otimes \mathsf{S}^k A^*$.

Consider the subbundle $\mathsf{S}_0^k A^*$ of $\mathsf{S}^k A^*$ consisting of the traceless tensors, i.e. such tensors $\xi \in \mathsf{S}^k A^*$ that $\operatorname{cotr}^s \xi = 0$. $\mathsf{S}_0^k A^*$ is then a $SO(n)$–irreducible subbundle of $\mathsf{S}^k A^*$.

Define the operator

$$\pi_{\operatorname{tr}} = \frac{1}{n+k-1} \operatorname{cotr}^s \circ \operatorname{tr} \quad \text{on} \quad A^* \otimes \mathsf{S}_0^k A^*.$$

and the three linear mappings:

$$\pi_1^s, \pi_2^s, \pi_3^s : A^* \otimes \mathsf{S}_0^k A^* \longrightarrow A^* \otimes \mathsf{S}_0^k A^*$$

by

$$\pi_1^s = \mathrm{Sym} \circ (\mathrm{id} - \pi_{\mathrm{tr}}), \qquad \pi_2^s = \mathrm{id} - \pi_1^s - \pi_3^s, \qquad \pi_3^s = \pi_{\mathrm{tr}}.$$

and next, the tree differential operators

$$P_j^s = \pi_j^s \circ \nabla^* : \mathsf{S}_0^k A^{*k} \longrightarrow A^* \otimes \mathsf{S}_0^k A^*, \qquad j \in \{1, 2, 3\}.$$

Theorem 7. ([7]) *$\pi_1^s, \pi_2^s, \pi_3^s$ are projections and $A^* \otimes \mathsf{S}_0^k A^*$ splits onto the direct (in fact, orthogonal) sum of $SO(n)$–invariant subspaces:*

$$A^* \otimes \mathsf{S}_0^k A^* = \mathrm{Im}\,\pi_1^s \oplus \mathrm{Im}\,\pi_2^s \oplus \mathrm{Im}\,\pi_3^s. \tag{19}$$

The covarant derivative ∇^s acting in the bundle $\mathsf{S}_0^k A^*$ splits onto the three operators

$$\nabla^s = P_1^s + P_2^s + P_3^s \tag{20}$$

of form

$$P_1^s = \frac{1}{k+1}\left(d^s + \frac{2}{n+k-1} g \odot d^{s*}\right),$$

$$P_2^s = \nabla^* - \frac{1}{k+1} d^s - \frac{2}{(n+k-1)(k+1)} g \odot d^{s*} + \frac{1}{n+k-1} \operatorname{cotr}^s d^{s*},$$

$$P_3^s = \frac{-1}{n+k-1} \operatorname{cotr}^s \circ d^{s*}$$

Remark 2.4. For $n \geqslant 5$ the orthogonal subspaces in (19) are irreducible, so, P_1^s, P_2^s, P_3^s in (20) are gradients. If $n = 4$, P_2^s splits further on two $SO(n)$-gradients. For $n = 3$ the decomposition into irreducible parts is given by the Clebsch-Gordon formula.

Remark 2.5. Notice that in the case $k = 1$ the splittings of ∇^a and ∇^s coincide up to the order of terms and then $P_1^s = P_2^a, P_2^s = P_1^a, P_3^s = P_3^a$.

Remark 2.6. Notice also that P_1^s is a symmetric counterpart of the Cauchy-Ahlfors operator described above. Finally notice that similarly as in the skew-symmetric case P_1^s is the only elliptic of the three considered gradients.

Now we are ready to pass to the second problem from our list.

2.5. System of natural boundary conditions

The method was suggested in 2004 by T.P. Branson and A. Pierzchalski and described in the unpublished manuscript (Tom Branson passed away in 2006). The method was next described in the W. Kozłowski and A,Pierzchalski paper [30] or in a more detailed version in A. Kimaczyńska and A. Pierzchalski [27]. Consider the operators of form

$$G^* G$$

where G is a gradient but the method is sufficiently general and fully applicable to other differential operators not necessarily being the gradients.

To describe our method we will need some integral formulas.

Recall that if ∇ starts from an irreducible bundle and if we split our target bundle onto irreducible orthogonal summands and we get orthogonal splitting

$$\nabla\alpha = G_1\alpha + \cdots + G_s\alpha.$$

A fundamental for the further construction is the observation by B. Orsted an A. Pierzchalski [34] that for each gradient $G = G_i$, $i = 1, \ldots, s$ which will be in fact a (Stein - Weiss) gradient we have the same universal integral formula for each gradient:

Theorem 8. ([34])

$$(G^*G\alpha, \beta) - (\alpha, G^*G\beta) = -\int_{\partial M} (\langle\alpha, i_\nu G\beta\rangle - \langle i_\nu G\alpha, \beta\rangle)\Omega_{\partial M} \tag{21}$$

where

$$(\cdot, \cdot) = \int_M \langle\cdot, \cdot\rangle \, \Omega_M$$

is the global scalar product.

Now we are ready to introduce our systems of natural boundary conditions .

First of all, the constructed condition should assert the self-adjointness of G^*G in the subspace of sections satisfying this condition. Otherwards, the integral on the right hand side of (21) should vanish. Roughly speaking, the trick here is to complete conditions in such a way that – from one side – they should be "not to weak" (in order "not to lose the uniqueness") and – from the other side – they should be "not to strong" (in order "not to lose the existence").

The next step is that at the boundary, the action of the special orthogonal group SO(n) is replaced by the action of of its subgroup - in fact isomorphic to SO($n-1$) - and consisting of transformations that keep the normal vector invariant. In a consequence, our up to now irreducible bundle splits at the boundary under the action of SO($n-1$) onto, say s, orthogonal, irreducible and invariant subbundles. Denote by $p_1, \ldots, p_s$ the projections defined by the splitting.

Decompose both α and $i_\nu G\alpha$ by taking their compositions with the projections p_i, $i = 1, \ldots, s$.

We get then that at the boundary:

$$\alpha = p_1\alpha + \cdots + p_s\alpha$$

and, similarly

$$i_\nu G\alpha = p_1 i_\nu G\alpha + \cdots + p_s i_\nu G\alpha$$

An analogous splittings are obtained for α replaced by β.

As a result, we get the following decomposition for the scalar products that appear in the integrand:

$$\begin{aligned}\langle\alpha, i_\nu G\beta\rangle = & \\ & + \langle p_1\alpha, p_1 i_\nu G\beta\rangle \\ & + \langle p_2\alpha, p_2 i_\nu G\beta\rangle \\ & \ldots \\ & + \langle p_s\alpha, p_s i_\nu G\beta\rangle \,.\end{aligned}$$

The right hand side of the last equality can be written (symbolically) in a form of a two column matrix

$$\begin{bmatrix} p_1\alpha & p_1 i_\nu G\beta \\ p_2\alpha & p_2 i_\nu G\beta \\ \vdots & \vdots \\ p_s\alpha & p_s i_\nu G\beta \end{bmatrix}$$

.

A natural boundary condition will be obtained then by the demand that exactly one term of each row of the matrix is equal to zero.

We get that way 2^s natural boundary conditions, namely, by exhausting all the possibilities of prescribing the zero value to exactly one term in each row.

Look at the two peripheral cases of our construction:

$$\begin{bmatrix} 0 & * \\ 0 & * \\ \vdots & \vdots \\ 0 & * \end{bmatrix}, \dots, \begin{bmatrix} 0 & * \\ * & 0 \\ \vdots & \vdots \\ 0 & * \end{bmatrix}, \dots, \begin{bmatrix} * & 0 \\ * & 0 \\ \vdots & \vdots \\ * & 0 \end{bmatrix}$$

The first matrix defines the condition: $p_1\alpha = 0$, $p_2\alpha = 0$, $\dots, p_s\alpha = 0$ on ∂M or, equivalently, $\alpha = 0$ on ∂M. It simply coincides with the Dirichlet type condition.

The last matrix defines the condition $p_1 i_\nu G\alpha = 0$, $p_2 i_\nu G\alpha = 0$, $\dots$,$p_s i_\nu G\alpha = 0$ on ∂M or, equivalently, $i_\nu G\alpha = 0$ na ∂M, It simply coincides with the Neumann type condition.

These two conditions together with the remained $2^s - 2$ ones compose the set of 2^s *natural boundary conditions* under which the operator G^*G is self-adjoint with respect to the global scalar product.

Example 2.7. *Consider the simplest case of the usual gradient operator acting on functions on M treating as sections of the trivial bundle $M \times \mathbb{R} \to TM$. We have then*

$$\text{grad} : M \times \mathbb{R} \to TM \tag{22}$$

where TM is the tangent bundle. Since the bundles $M \times \mathbb{R}$ and TM are both irreducible the operator (22) is a gradient in our sense. Its formal adjoint is the negative divergence operator

$$(\text{grad})^* = -\text{div}$$

So the composition -div grad *is the negative classical Laplacian Δ on functions. The known Stokes formula says in this case that*

$$\int_M \Delta\ f\ g\ \Omega_M - \int_M f\ \Delta\ g = -\int_{\partial M} (f\ \nabla_\nu g - \nabla_\nu f\ g)\Omega_{\partial M}.$$

Since the bundle $M \otimes R$ is irreducible, we have a one row matrix

$$\begin{bmatrix} * & * \end{bmatrix}.$$

So, s=1, and the the natural boundary conditions take the form:

$$\begin{bmatrix} 0 & * \end{bmatrix} \text{or} \begin{bmatrix} * & 0 \end{bmatrix},$$

or, explicitly,

$$f = 0 \quad \text{on} \quad \partial M \quad \text{or} \quad \nabla_\nu f = 0 \quad \text{on} \quad \partial M.$$

That way we get the well known: Dirichlet or Neumann boundary conditions, respectively.

The presented simple example show the naturality of the construction of the system of natural boundary conditions (SNBC).

The function f in the example can be treated as form of degree zero (skew-symmetric and the symmetric at the same time).

For forms of higher degree the number od summands at the boundary depends on the symmetry type.

Each skew-symmetric form of degree $k = 1 \dots n$ (i.e. greater than zero) decomposes at the boundary onto two summands: its tangent and the normal parts. So $s = 2$ independently on the degree k. There are $2^2 = 4$ natural boundary conditions in the bundle of skew-symmetric forms.

Each symmetric form of degree $k = 1 \dots n$ decomposes at the boundary onto $k + 1$ summands described below. So $s = k + 1$ and - in contrast to the skew-symmetric case - depends on the degree k There are there 2^{k+1} natural boundary conditions in the bundle of symmetric forms of degree k.

In the general case the number of summands is determined by the branching rule from the whole group $\mathrm{SO}(n)$ to its subgroup $\mathrm{SO}(n-1)$ and my be found e.g. in [20].

Coming back to the general case we can say that our systems of just constructed boundary conditions is in some intuitively natural and complete. Indeed, its naturality comes from the fact that in its construction the representations of $\mathrm{SO}(n)$ - the group naturally related to the Riemannian structure were used. The completeness comes from the fact that the considered representations decomposes there completely, i.e. decomposes onto the full list of its irreducible subrepresentations.

More interesting and important is yet the problem of the ellipticity (at the boundary) of the each particular condition from (SNBC).

The boundary value problem for an arbitrary Riemannian manifold with boundary for S^*S was studied by B. Ørsted and A. Pierzchalski in [34]. Three boundary conditions analogous to the conditions studied by Weyl were investigated there. The asymptotic distribution of the eigenvalues for the Ahlfors Laplacian S^*S was derived there.

The system (SNBC) consists - in the case of the bundle of skew-symmetric forms of degree k - of four natural boundary conditions. The first three coincide with the conditions considered by Weyl. The forth one is new. It completes the list of the previous three ones with a new surprising symmetry.

Let us describe shortly the situation. For arbitrary k a skew-symmetric k–form splits (at the boundary) into two summands: its tangent and the normal parts:

$$\omega = \omega^T + \omega^N.$$

In a consequence the bundle of skew-symmetric k–forms splits at the boundary (under the action of $\mathrm{SO}(n)$) onto a direct sum of two subbundles.

So $s = 2$ and there are four $(= 2^2)$ natural boundary conditions [30]:

Dirichlet boundary condition $(\mathcal{D})$:

$$\omega^T = 0 \quad and \quad \omega^N = 0 \quad \text{on} \quad \partial M.$$

Absolute boundary condition $(\mathcal{A})$:

$$\omega^N = 0 \quad and \quad (d\omega)^N = 0 \quad \text{on} \quad \partial M.$$

Relative boundary condition $(\mathcal{R})$:

$$(\delta\omega)^T = 0 \quad and \quad \omega^T = 0 \quad \text{on} \quad \partial M.$$

Neumann boundary condition ($\mathcal{N}$):

$$(\delta\omega)^T = 0 \quad and \quad (d\omega)^N = 0 \quad \text{on} \quad \partial M.$$

Notice the following surprising symmetry in the system $\{\mathcal{D}, \mathcal{A}, \mathcal{R}, \mathcal{N}\}$ with respect to the Hodge $*$–operator.

Theorem 9. ([30], [46]) *Each of the boundary conditions $\mathcal{D}$, $\mathcal{N}$ is star invariant in the sense that a k–form φ satisfies a condition if and only if the $n-k$–form $*\varphi$ satisfies the condition. The set of conditions $\{\mathcal{A}, \mathcal{R}\}$ is the star invariant in the sense that a k–form φ satisfies the condition if and only if the $(n-k)$–form $*\varphi$ satisfies the other one condition.*

The ellipticity of the conditions analogous to $\{\mathcal{D}, \mathcal{A}, \mathcal{R},\}$ was proved by B. Ørsted and A. Pierzchalski in 1996 (before the natural system of boundary conditions was formulated). The proof was published in [34] for the case of a general compact Riemannian manifold with a smooth boundary. The proof of ellipticity constructed there for each of three conditions separately can also be extended onto the fourth natural condition $\{\mathcal{N}\}$. In a consequence the updated version of Theorem 4.2 from [34] can now be formulate as follows.

Theorem 10. ([34]) *With respect to either of the four natural boundary conditions: $\{\mathcal{D}, \mathcal{A}, \mathcal{R}, \mathcal{N}\}$ the weighted Laplacian $L = \Delta_{ab}$ is self–adjoint and elliptic. In particular, L has a complete orthonormal system of eigenforms $\alpha_1, \alpha_2, \ldots$ for each boundary condition: $L\alpha_k = \lambda_k\alpha_k$, with α_k of class C^∞ and satisfying the boundary condition in question. Here the eigenvalues $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ grow exponentially.*

The obtained system of boundary conditions have been also successfully tested by W. Kozłowski and A. Pierzchalski in [30] and earlier for the Dirichlet condition by W. Kozłowski in his PhD paper [29]. They considered there weighted Laplacian acting on skew skew-symmetric forms of any degree k in the euclidean ball in $\mathbb{R}^n$. The boundary value problem for all the four boundary conditions $\{\mathcal{D}, \mathcal{A}, \mathcal{R}, \mathcal{N}\}$ and the polynomial boundary data was solved completely. Moreover a construction of the explicit solution for each of the boundary condition was given.

Let us pass to the case of operators acting on the bundle of symmetric forms of arbitrary degree k. This case is more difficult than the previous one and so, by nature, much less investigated in the literature. Recently, it has been changing. The symmetric forms are studied more and more intensively and more and more papers has been appearing on this subject. Let us only mention the recent paper [21] on Killing and conformal Killing tensors, i.e. the symmetric tensors from the kernel of a $\mathrm{SO}(n)$–gradient d_0^s where d_0^s is the symmetric and trace free part of the symmetric derivative.

The first difference between the bundles of skew-symmetric and symmetric forms is that the other is of infinite rank. The next, and in our case more essential difference is that the bundle of symmetric tensors of degree k splits onto $k+1$ summands at the boundary, so $s = k+1$, and – in contrast to the skew-symmetric case – the number of summands in the splitting depends on the degree of forms. In a consequence there are

2^{k+1} natural boundary conditions for the bundle of symmetric tensors of degree k. For big k this gives a huge number of conditions.

In the recent paper by A. Kimaczyńska and A. Pierzchalski in [27], the following results were stated:

Theorem 11. (**[27]**) *All the 2^{k+1} natural boundary conditions for the second order elliptic operator divgrad in the bundle of symmetric k–forms are elliptic*

Let us remark that by applying our original method we were able to prove the ellipticity of all the boundary conditions simultaneously. This contrast to the until now known proofs of the ellipticity for other bundles where methods of proofs are constructed to each the boundary condition separately.

A consequence of the ellipticity of boundary conditions is the following fact:

Theorem 12. (**[26]**) *For each of all considered 2^{k+1} boundary conditions there exists a sequence (ϑ_n), $n = 1, \dots$ of smooth sections of the bundle of symmetric k - tensors on M such that:*

a) (ϑ_n) is a complete orthonormal system in L^2 of eigenvectors

$$div\, grad\, \vartheta_n = \lambda_n \vartheta_n,$$

b) the forms ϑ_n satisfy the boundary condition
c) The eigenvalues λ_n are real and

$$\lim_{n\to\infty} \lambda_n = -\infty.$$

Regarding the system of natural boundary condition (SNBC), in some specific cases it is possible to go further and provide the necessary and sufficient conditions for the existence and uniqueness of solutions, or even further, and give the construction of these solutions in specific cases. Let us briefly discuss this part of the activity.

Let M be a n–dimensional Riemannian manifold with a non-empty boundary ∂M. Let g be a Riemannian metric on M and let ∇ be the Levi-Civita covariant derivative of g (we extend g to the whole tensor bundle). In the case of skew-symmetric forms of any degree $k = 1, \dots, n$ on the manifold M the only elliptic gradient - also denoted by S - composed with its formal adjoint operator S^* gives a strong elliptic operator of the second order S^*S. In the case of $k = 1$, this operator is called the Ahlfors-Laplace operator or just the Ahlfors Laplacian.

Applying the Weitzenböck formula we get a very important formula.

$$S^*S = \frac{1}{2}d^*d + \frac{n-1}{n}dd^* - \mathrm{ric}, \tag{23}$$

where ric is an operator of order zero. More precisely, ric is given by Ricci action on 1-forms: $(\mathrm{ric}\alpha)_i = \mathrm{ric}_i^j \alpha_j$. Two versions of this formula: for forms and for vector fields can be found, e.g., in my work ([43] Theorem 2), but in fact the formula was known to the geometers much earlier.

The importance of the formula comes from the fact that it combines all the three gradients from the decomposition (8) with the Ricci tensor representing the geometry of M. The formula also allows to determine the spectrum of S^*S in many special cases e.g. for the Einstein manifolds, and also to estimate the constant of quasiconformality af ant any deformation on M in dependence on the Ricci tensor. The results of research in this filed are included in my publication in *manuscripta mathematica* [44].

Another important advantage of the formula is that it shows that - with an exactness to the zero order component (the last component in the decomposition (23)) the Ahlfors-Laplacian S^*S is a particular case case of the weighted form Laplacian, i.e., of the operator of form

$$\Delta_{ab} = a\delta d + bd\delta, \tag{24}$$

where a, b are positive reals called *weights*.

Moreover, the weighted form Laplace operator is well defined on any skew-symmetric form of arbitrary degree $k \geqslant 0$.

For $k = 1$ and the bounded domain in $\mathbb{R}^3$ the behavior on boundary for such an operator was investigated by H. Weyl at the beginning of the 20th century, in the special case of weights: $a = \frac{1}{2}$ and $b = \frac{2}{3}$. Weyl studied the boundary problem in this case for three physically natural boundary conditions. He also determined the distributions of the eigenvalues of $\Delta_{\frac{1}{2}\frac{2}{3}}$ for each of these conditions (cf. [51]).

For $k = 1$ and the bounded domain in $\mathbb{R}^n$ and the weights $a = \frac{1}{n}$ and $b = \frac{n-1}{n}$, the operator $\Delta_{\frac{1}{n}\frac{n-1}{n}}$ was investigated by L.V. Ahlfors in a series of papers from the 1970s [1], [2], [3]. In particular, he solved the boundary value problem for the Dirichlet boundary condition and the analogous operator in the hyperbolic sphere in $\mathbb{R}^n$. He brilliantly used the property of conformal invariance of the considered operator and the fact that the group of conformal transformations of the sphere acts transitively and that it is the group of isometries with respect to the hyperbolic metric. Consequently, it was enough to then determine the value of the solution in the center of the sphere using the Poisson type formula and "translate" it to any point using the appropriate isometry. The Dirichlet boundary problem for the Euclidean ball in $\mathbb{R}^n$ was more difficult. The Ahlfors method could not be used because the isometric group isometries of the Euclidean ball does not act transitively. However, H. Riemann, in the work of [47], in analogy to the well known procedure for the Dirichlet boundary problem for Laplace operator on functions, applied the fact that every L^2-function on the sphere may be decomposed into a series of spherical harmonics. Riemann decomposed - using representation theory - the space of vector fields (or, equivalently 1-forms) into appropriately selected $SO(n)$–invariant subspace, and then he found a convenient base in each of them. These bases allowed him to construct a solution.

For any k, the Euclidean sphere in $\mathbb{R}^n$ and weighted Laplacian (24) acting on the bundle of skew-symmetric forms with polynomial coefficients, the boundary value problems were investigated jointly by W. Kozłowski me ([30]). All the boundary problems from the list of the (SNBC) were investigated. The system contains four boundary conditions in this case. The discussed above: Dirichlet, absolute, relative and Neumann one $\{\mathcal{D}, \mathcal{A}, \mathcal{R}, \mathcal{N}\}$. The first three boundary conditions appeared in differential geometry earlier under these names, e.g, when studying the asymptotic expansion of the heat kernel of the Laplace type operators (see, for example, [14] or [15]). The fourth boundary condition is new and has been studied for the first time. The Dirichlet boundary condition $\{\mathcal{D}\}$ was studied in the Ph.D. dissertation by W. Kozłowski [29]. in this particular case and the forms of any degree k, the theorem on the existence and uniqueness of solutions holds without any exceptions. Excluding some exceptional cases it also holds for the pair of relative and absolute conditions. In each of these exceptional cases, the necessary and sufficient conditions for the existence of solutions are given and the uniqueness of the solutions is discussed. The most interesting is the fourth condition $\mathcal{N}$. The existence of solutions requires some additional assumptions on the boundary data. However, also in this case necessary and sufficient conditions for the existence of solutions are given. The proofs of the theorems on existence are constructive so the solutions are given explicitly for each given boundary data.

Let us quote here some theorems from [30] concerning the conditions $\mathcal{R}$ and $\mathcal{A}$. Remind that any form decomposes at the boundary - according to the formula **??** - for its tangent and normal parts. The form is called *tangent* when its has no normal part and is called *normal* when its has no tangent part. The unit ball in $\mathbb{R}^n$ is denoted by B and its boundary, i.e., the unit sphere, by Σ. All the forms considered here are forms in $\mathbb{R}^n$ with polynomial coefficients.

Theorem 13. ([**30**] Relative boundary condition) *Let $0 \leqslant k < n$. For any tangent forms: ω of degree $k-1$ and η of degree k there exists a unique $\varphi \in \Lambda^k$ such that $\Delta_{ab}\varphi = 0$ in B with*

$$(d^*\varphi)^{\mathrm{T}} = \omega \quad \varphi^{\mathrm{T}} = \eta \quad \text{on } \Sigma.$$

Theorem 14. ([**30**]Absolute boundary condition) *Let $0 < k \leqslant n$. For any normal forms ω of degree $k+1$ and η of degree k there exists a unique $\varphi \in \Lambda^k$ such that $\Delta_{ab}\varphi = 0$ and*

$$(d\varphi)^{\mathrm{N}} = \omega \quad \varphi^{\mathrm{N}} = \eta \quad \text{on } \Sigma.$$

In [30] the necessary and sufficient conditions for the existence of solutions are formulated for each of the exceptional cases: $k = n$, when $\mathcal{R}$ is considered or $k = 0$ when $\mathcal{A}$ is considered. Also, in each of the two cases the uniqueness of the solutions is discussed.

Note (cf.[30]) that investigation of the pair of conditions $\{\mathcal{AR}\}$ can be reduced to the study of any one them thanks to the following property, which holds not only for the case of the Euclidean sphere, but - in general - for any Riemannian manifold M with a smooth boundary ∂M.

Theorem 15. (**[30]**) *Let ω and η be a continuous tangential $(k-1)$–form and k–form, respectively, defined on ∂M. A k–form ψ is a solution to the relative boundary condition $\Delta_{ab}\varphi = 0$ in M with $(d^*\varphi)^{\mathrm{T}} = \omega$ and $\varphi^{\mathrm{T}} = \eta$ on ∂M if and only if $(n-k)$–form $\star\psi$ is a solution to the absolute boundary condition $\Delta_{ba}\varphi = 0$ in M with $(d\varphi)^{\mathrm{N}} = (-1)^{k+1} \star \omega$ and $\varphi^{\mathrm{N}} = \star\eta$ on ∂M.*

The case of the fourth natural boundary condition, the Neumann one $\mathcal{N}$ seems to be of particular interests.

$$\Delta_{ab}\varphi = 0 \;\text{ na }\; B \;\text{ oraz }\; (d^*\varphi)^T = \omega, \; (d\varphi)^N = \eta \;\text{ na }\; \partial M.$$

This boundary condition has not been discussed earlier and in [30] is investigated for the first time. It is elliptic. However, for getting the existence of solutions, some additional conditions have to be imposed on ω and η. These conditions are necessary and sufficient and they are formulated in theorem 4.10 from [30]. It would be difficult to state them explicitly here since they are rather technical and the formulation would require some preparatory material to be introduced just for the need of only this one particular theorem. The problem of uniqueness is also discussed there.

Finally, let us add, that the natural system of boundary conditions (SNBC) was intensively studied for the ellipticity, jointly by T. Branson and me. This took place, in particular, during my stay (as *visiting associate professor*) at the University of Iowa in the period of 2003/2004. The conjuncture that for any elliptic gradient the (SNBC) consists of only elliptic boundary conditions seems to be very bold, but there are no counter-examples and all the investigated special cases confirm the conjuncture. Our nearly two-year effort to find a proof to the hypothesis did not bring results. The premature death of T. Branson in 2006 interrupted our collaboration. The proof of the general case seems to be rather difficult. However, the results of the described above joint work with B. Ørsted, later with W. Kozłowski and recently with A. Kimaczyńska confirm the hypothesis in three important particular cases.

2.6. The geometry of gradients

In his paper on elasticity ([51]) H.Weyl used the boundary condition of vanishing divergence and vanishing tangential part of the vector field on the boundary of a domain in $\mathbb{R}^3$. In our paper [34] on the boundary behavior of the Ahlfors Laplacian S^*S and practically also its generalization: the weighted Laplacian Δ_{ab} defined in (24), three different, analogous to that of Weyl - boundary conditions under the names: $\mathcal{D}$, $\mathcal{N}$ and $\mathcal{E}$ are considered. One of them $\mathcal{E}$ is the exact version of the Weyl elasticity condition in much more general situation: a manifold instead of domain in $\mathbb{R}^3$, differential forms of any degree k instead of vector fields. Also the Weyl's operator is a particular version with $a = \frac{2}{3}$ and $b = \frac{1}{2}$ of our weighted Laplacian Δ_{ab}. The condition $\mathfrak{E}$ was later recognized as the relative one $\mathfrak{R}$ from the list (SNBC) and under such the name it occurs in our Theorem 10 above.

One of the consequence of this theorem and in particular of the existence of the complete system: $\alpha_1, \alpha_2, \ldots,$ $L\alpha_k = \lambda_k\alpha_k$, of smooth eigenforms α_k satisfying the condition $\mathfrak{R}$ is that we can consider the heat semigroup $\exp(-\mathrm{tL})$ which is a kernel operator with a smooth kernel

$$H(t,x,y) = \sum_{k=1}^{\infty} e^{-t\lambda_k}\alpha_k(x) \otimes \alpha_k(y) \tag{25}$$

and next its trace

$$tr\ exp(-tL) = \sum_{k=1}^{\infty} e^{-t\lambda_k} = \int_M H(t,x,x)\Omega_M. \tag{26}$$

It follows from the general principle (see e.g. [18]) that (26) has an asymptotic expansion

$$tr\ exp(-tL) \sim \sum_{k=1}^{\infty} a_i t^{i-n}2, \quad t \searrow 0.$$

Theorem 16. **([34])** *Let L be the self-adjoint extension of the Ahlfors Laplacian with the boundary condition $\mathfrak{R}$ on M. Then the small-time asymptotics of the heat kernel have two first terms as follows:*

$$\begin{aligned} tr\ exp(-tL) \sim & (4\pi t)^{\frac{n}{2}} vol(M)[(n-1)a^{-\frac{n}{2}} + b^{\frac{n}{2}})] \\ & -\frac{1}{4}(4\pi t)^{-\frac{(n-1)}{2}} vol(\partial M)[(n-3)a^{-\frac{n-1}{2}} + b^{-n(n-1)}] \end{aligned} \tag{27}$$

Next, by the Tauberian theorem, the asymptotic distribution of eigenvalues was deduced, generalizing and sharpening that way the main theorem in

Theorem 17. **([34])** *Let $N(\lambda)$ denote the number of eigenvalues for L less then λ. Then, for our boundary condition $\mathfrak{R}$, the first terms in the asymptotic expansion for $N(\lambda)$ is*

$$N(\lambda) \sim \frac{(4\pi)^{-n/2}}{\Gamma(n/2+1)} vol(M) \cdot [a^{-n/2} + (n-1)b^{-n/2}] \cdot \lambda^{n/2}$$

If $n = 3$, $a = \frac{2}{3}$ i $b = \frac{1}{2}$, for the coefficient $(4\pi)^{-\frac{3}{2}}/\Gamma(\frac{3}{2}+1)$ we get the exact value obtained by Weyl: $\frac{\pi^{-2}}{6}$.

I conducted, together with T. Branson, P. B. Gilkey and B. Ørsted, investigation on asymptotic expansions for Laplace type operators. The results have been published in [14] and [15]. In both papers, a second order operator of form

$$P = ad^*d + bdd^* - \epsilon\rho, \tag{28}$$

is investigated, where ρ is any zero order selfadjoint operator. The operator P is a direct generalization of the Ahlfors Laplacian S^*S (23) and it is an elliptic and selfadjoint operator. In the discussed papers we determined first four coefficients of the asymptotic expansions of the heat kernel for this operator. In the first paper: for compact Riemannian manifold without boundary. In the second: for Riemannian manifold with boundary and for the two boundary conditions: relative $\mathcal{R}$ and absolute $\mathcal{A}$. Coefficients of the asymptotic expansion are fundamental tool for spectral theory of differential operators, especially for manifold geometry. These coefficients play important role when investigating so called inverse problem of spectral geometry. In this subject, the question is to reconstruct the geometry based on information encoded in the investigating operators: in their spectra or in the asymptotic expansions of the heat kernels. The latter ones, in case of manifolds with boundary, carry an information about the geometry of both the manifold and the boundary. The method used in our papers is analogical to the method used in the paper of T. Branson, P. B. Gilkey and S. Fulling [13], but completely different from the one used by B. Ørsted and myself in [34]. Moreover, in contrary to our method, the method used in [13] does not cover Dirichlet condition $\mathcal{D}$.

Finally, we will present some results from gradient geometry on foliated manifolds.

Properties of Riemannian manifolds gradients and their role were inducing the questions on the importance of gradients to investigate other foliated manifolds geometric structures. It occurred to us, that also in these cases, the application of gradients leads to interesting results.

We noticed, that the foliation of the class SL(q) defined and investigated by Ph. Tondeur [49] (however in a different context) is a good class from a gradient theory point of view in a sense that it is conform with restriction operation. Moreover it is quite voluminous (codimension of Lie group SL(q) in linear group GL(q) is one) and it is the broadest foliation class on which coherent gradients theory can be exercised without losing their basic geometric attributes. In earlier works of other authors in regards to restriction of natural differential operators to foliations, standard assumption (see for example [4] or [5] of A. Alvarez Lopez, Y. A. Kordyukov) was that the foliation in question are Riemannian. This assumption - in comparison to ours - is very strong. But yet our conclusions are very similar. As an example, let us take the derived by us the

Weitzenböck formula for foliations, one of the most important theorem in differential geometry relating two Laplacians the Hodge and the Bochner ones acting on forms of any degree k and with values in any Riemannian vector bundle with metrics h. Upper index $\mathcal{F}$ at operator symbol denotes that the operator is related to leaves of a foliation.

Theorem 18. ([9]) *Assume that (M,g) is a Riemannian manifold, (E,h)–a Riemannian bundle over M and $\mathcal{F}$ is an $SL(q)$–foliation. Then, for any $\sigma \in \Lambda^k T\mathcal{F}^* \otimes E$,*

$$\Delta^{\mathcal{F}}\sigma = -trace^{\mathcal{F}}\left(\nabla^{\mathcal{F}}\right)^2\sigma + S^{\mathcal{F}}(\sigma),$$

where

$$S^{\mathcal{F}}(\sigma)(X_1,\ldots,X_k) = \sum_{s=1}^{p}\sum_{j=1}^{k}(-1)^j\left(R^{\mathcal{F}}(e_s,X_j)\,\sigma\right)\left(e_s,X_1,\ldots,\hat{X}_j,\ldots,X_k\right)$$

oraz

$$R^{\mathcal{F}}(e_s,X_j) = -\nabla^{\mathcal{F}}_{e_s}\nabla^{\mathcal{F}}_{X_j} + \nabla^{\mathcal{F}}_{X_j}\nabla^{\mathcal{F}}_{e_s} + \nabla^{\mathcal{F}}_{[e_s,X_j]},$$

for $X_1,\ldots,X_k \in T\mathcal{F}$, and a local orthonormal frame $(e_1,\ldots,e_p)$ of vectors tangent to the leaves $\mathcal{F}$.

In a cooperation with A. Bartoszek and J. Kalina ([9]), we proved that on any SL(q)–foliation there exists a special, local coordinate system that is convenient tool in foliated manifold geometry. This coordinate system enabled to identify many interesting properties of geometric gradients, in particular, it was very useful in the proof of the last theorem.

A year later, in the paper [10], we investigated also gradients and the Dirac operator on SL(q)–foliation. A key observation is, that operators conjugate to operators restricted to foliations can be expressed by an integral over full manifold M, the and integrand have the standard form. Moreover, the described above Theorem refO-P being the key theorem when the boundary problems are investigated holds also here for the $\mathcal{F}$–relative gradients in an analogous version

Let ξ_α and $\tilde{\xi}_\beta$ be any irreducible subbundles of ξ and $\tilde{\xi}$, respectively, and $\nabla^{\mathcal{F}\alpha\beta}$: $\xi_\alpha \to \tilde{\xi}_\beta$, the relative gradient.

Theorem 19. ([10]) *For any $s \in \xi_\alpha$ and $u \in \tilde{\xi}_\beta$.*

$$\int_M \left\langle \nabla^{\mathcal{F}\alpha\beta}s, u\right\rangle = \int_M \left\langle s, -\pi_\alpha tr^{\mathcal{F}}_{12}\nabla^{\mathcal{F}}\iota_\beta u\right\rangle.$$

Otherwards, the operator $\nabla^{\mathcal{F}\alpha\beta*}$ formally adjoint (on M) to the operator $\nabla^{\mathcal{F}\alpha\beta}$ is of form

$$\nabla^{\mathcal{F}\alpha\beta*} = -\pi_\alpha tr^{\mathcal{F}}_{12}\nabla^{\mathcal{F}}\iota_\beta.$$

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- 1975 PhD in Mathematics from the Polish Academy of Sciences, doctoral thesis "Algebraic criterion for quasiconformal equivalence of Riemannian manifolds"[35] and [36].
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Employment

- 1971–1974 Doctoral Study, Institute of Mathematics, Polish Academy of Sciences
- 1974–1975 Assistant Professor, Institute of Mathematics, Polish Academy of Sciences, Warsaw
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Selected Scientific Interests

- Differential Geometry
- Global Analysis
- Geometry of Differential Operators

Selected Invited Academic Visits

- Odense University, Danmark (1984, 1987, 1988, 1989, 1990, 1994)

- Copenhagen University, Danmark (1996, 1998)
- Cagliari University, Italy (1998)
- University of Iowa, USA (1991-two semesters) visiting associate professor.
- Washington University in St. Louis, USA (1991)
- Oregon University, USA (1991)
- Odense University, Danmark (2000)
- University of Iowa, USA (2003-04 three semesters), visiting associate professor.
- Kazan State University, Russia (2011)
- Aarhus University, Denmark (2013, 2015, 2016, 2018)

Selected Invited seminars

- 2007 Czestochowa Technical University
- 2012 University of Warmia and Mazury, Olsztyn
- 2012 Jagiellinian University, Kraków
- 2015 Bialystok Technical University
- 2015 Polish Academy Of Sciences, Warsaw, Seminar on Geometric Methods in Physics
- 2016 Adam Mickiewicz University, Poznań
- 2017 Warsaw University, Algebraic Topology Seminar

Selected Conferences

- 2007 Midwest Geometry Conference at the University of Iowa, Iowa City, IA USA, 18 - 20 May 2007
- 2009 XV Conference On Analitic Functions And Related Topics, *Weitzenböck Formula for $SL(q)$–foliations*, Chelm, June 2009, plenary lecture
- 2011 XVI Conference On Analitic Functions And Related Topics, *Gradients on leaves of a foliation*, Chelm, June 2011, plenary lecture
- 2011 International Conference On Current Problems In Science And Art, Kazań
- 2012 Differential Geometry, *Weitzenböck Formula on Lie algebroids*, Będlewo, June 2012
- 2013 Nonlinearities, *On conjugate submersions*, Małe Ciche, June 2013
- 2013 Differential Geometry And Applications, *Algebra and geometry of natural differential operators*, Brno, August 2013
- 2013 Analiza i Topologia, *Laplacian in a symmetric bundle*, Gdańsk, September 2013
- 2013 Differential Geometry, *Gradients and Dirac operators on algebroids*, Kraków, November 2013
- 2014 Vector Distributions and Related Geometries, *Conjugate Submersions*, Warszawa, June 2014
- 2014 The Sixth Podlasie Conference on Mathematics, *Dirac operators on Lie algebroids*, Białystok, July 2014; Organizator sesji Operatory różniczkowe: algebra, geometria, reprezentacje
- 2015 Workshop on Dirac operators, quantum groups, and Lie algebroids Department of Mathematics Aarhus University, *Gradients and the Dirac operators on Lie algebroids*, Aarhus, March 2015
- 2015 3rd Conference on Dynamical Systems and Applicati, *On differential operators and geometric structures*, Łódź, April 2015
- 2016 Analysis Seminar, *Differential operators, geometry*, boundary conditions, Aarhus, May 2016

- 2017 Hypercomplex Seminar, *Elliptic boundary conditions for the operators of gradient and divergence in the bundle of symmetric tensors*, Będlewo, July 2017
- 2018 XIX-th International Conference On Analytic Functions And Related Topics, *Geometry of Laplace type operators*, Rzeszów, June 2018
- 2018 Glances at Manifolds 2018 Kraków, *Differential operators in tensor bundles for different geometries*, June 2018
- 2018 Analysis Seminar, *The elliptic boundary conditions for different bundles and geometries*, Aarhus, November 2018

Supervisor in PhD trials - completed

- Dorota Blachowska, University of Łódź (completed in 2003)
- Wojciech Kozłowski, Polish Academy of Sciences, Warsaw (completed in 2005)
- Szymon Walczak, University of Łódź (completed in 2005)
- Kamil Niedziałomski, University of Łódź (completed in 2008)
- Małgorzata Ciska, University of Łódź (completed in 2012)
- Anna Każmierczak, University of Łódź (completed in 2013)
- Agnieszka Klekot, University of Łódź (completed in 2015)
- Anna Kimaczyńska, University of Łódź (completed in 2017)

Supervisor in PhD trials - open

- Agnieszka Najberg, University of Łódź (open from 2013)

Refree in PhD Trials

1. Maciej Czarnecki, University of Łódź (2000)
2. Marek Badura, University of Łódź (2002)
3. Wojciech Kawa, University of Łódź (2005)
4. Agata Bezubik, Warsaw Technical University (2010)
5. Magdalena Lużyńczyk, University of Łódź (2015)
6. Tomasz Zawadzki, University of Łódź (2015)
7. Tomasz Kostrzewa, Warsaw Technical University (2019)

Educational achievements

- I was advisor in about 50 master thesis.
- I was advisor of 7 individual students.
- Additionally four of my students were awarded by the ministry stipend.
- Many of my students ware granted with the diploma with distinction.

List of publications

1) **A. Pierzchalski** On relations between linking hypersurface families in Euclidean spaces, Bull. Acad. Polon. Sci., XXI (1973), 977-981.
2) **A. Pierzchalski** Lebesgue measure on Riemannian differential spaces, Bull. Acad. Polon. Sci., **XXII** (1974), 1015-1020.
3) **A. Pierzchalski** Algebraic criterion of quasiconformality for Riemannian differiential spaces, Bull. Acad. Polon. Sci.,**XXIII** (1975), 305-313 (rozprawa doktorska).

4) **A. Pierzchalski**, *The k-module of level sets of differential mappings*, Scientific Communications of the Czechoslovakian-GDR-Polish School on Differential Geometry at Boszkowo (1978), Math. Inst. Polish Acad. Sci., Warsaw, 1979, 180-185.
5) **A. Pierzchalski** A variation of the modulus of submanifold families, Analytic Functions, Kozubnik, 1979, Lecture Notes in Math. 798, Springer-Verlag, Berlin-Heidelberg-New York, 1980, 261-268.
6) **A. Pierzchalski**, *Przestrzenie kontaktowe.* Szkoła Letnia Funkcji Analitycznych, Blażejewko 1982, Uniwersytet Łódzki, 1982, 91-112.
7) **A. Pierzchalski**, *On quasiconformal deformations on manifolds*, Romannian-Finnish Seminar on Complex Analysis, Proceedings, Part 1, Lecture Notes in Math. 1013, Springer-Velag, Berlin-Heidelberg-New York-Tokyo, 1983, 171-182.
8) **A. Pierzchalski**, *On quasiconformal deformations of manifolds and hypersurfaces*, Ber. Univ. Jyvaskyla Math. Inst., **28** (1984), 79-94.
9) **A. Pierzchalski**, *Quasiconformality of pseudo-conformal transformations and deformations of hyper-surfaces in* $\mathbb{C}^n$, Math. Scand., **59** (1986), 223-234.
10) **A. Pierzchalski** *Some differential operators connected with quasiconformal deformations on manifolds*, Partial Differential Equations, Banach Center Publ. **19**, PWN-Polish Scientific Publishers, Warszawa 1987, 205-212.
11) **A. Pierzchalski**, *Ricci curvature and quasiconformal deformations of a Riemannian manifold*, manuscripta mathematica **66** (1989), 113–127.
12) **J. Kalina, A. Pierzchalski** *Some differential operators in real and complex geometry* in Deformations of Mathematical Structures, 1989, Kluwer Academic Publishers, 3-28.
13) **B. Ørsted, A. Pierzchalski**, *The Ahlfors Laplacian on a Riemannian manifold*, Constantin Caratheodory: An International Tribute, 2, World Scientific, Teaneck, NJ, 1991, pp. 1021–1049.
14) **T.P. Branson, P.B. Gilkey, B. Ørsted, A. Pierzchalski**, *Heat equation asymptotics of a generalized Ahlfors Laplacian on a manifold with boundary*, in Operator Theory: Advances and Applications, **57**, Birkhauser Verlag, Basel, 1992, 1–13.
15) **A. Pierzchalski**, *Rachunek wariacyjny.* Leksykon Matematyczny, Wiedza Powszechna, Warszawa 1993.
16) **T.P. Branson, P.B. Gilkey, A. Pierzchalski**, *Heat equation asymptotics of elliptic operators with nonscalar leading symbol*, Math. Nachr. **166** (1994), 207–215.
17) **J. Kalina, B. Ørested, A. Pierzchalski, P. Walczak, G. Zang**, *Elliptic gradients and highest weights*, Bull. Polon. Acad. Sci. Ser. Math. **44** (1996), 511–519.
18) **B. Ørsted, A. Pierzchalski**, *The Ahlfors Laplacian on a Riemannian manifold with boundary*, Michigan Math. J. **43** (1) (1996), 99–122.
19) **A. Pierzchalski**, *Geometry of quasiconformal deformations of Riemannian manifolds*, (habilitation dissertation) Lodz University Press, 1997.
20) **J. Kalina, A. Pierzchalski, P. Walczak**, *Only one of the generalized gradients can be elliptic*, Ann. Polon. Math. **67** (1997), 111–120.
21) **T.P. Branson, A. Pierzchalski**, *Natural boundary conditions for gradients*, manuscript, 2004
22) **A. Pierzchalski, P. Walczak**, *Generalized gradients on foliated manifolds.* WMiI UŁ, preprint 2006/03.
23) **W. Kozłowski, A. Pierzchalski**, *Natural boundary value problems for weighted form Laplacians*, Ann. Sc. Norm. Sup. Pisa, **VII** (2008), 343–367.
24) **A. Bartoszek, J. Kalina, A. Pierzchalski**, *Weitzenböck formula for SL(q)-foliations.* Bull. Pol. Acad. Sci. Math. **58** (2010), no. 2, 179–188.

25) **A. Bartoszek, J. Kalina, A. Pierzchalski**, *Gradients for* SL(q)*–foliations.* J. Geom. Phys. **61** (2011), no. 12, 2410–2416.
26) **A. Pierzchalski**, *Natural boundary value problems for Ahlfors-Laplacian.* Proceedings of International Conference "Current Problems of Art and Science I ", Kazan University Press, Kazań, 2011, 132-137.
27) **A. Klekot, A. Pierzchalski**, *Weitzenböeck formula for vector valued forms.* WMiI UŁ, preprint 2011/15.
28) **B. Balcerzak, J. Kalina, A. Pierzchalski**, *Weitzenböck formula on Lie algebroids.* Bull. Pol. Acad. Sci. Math. **60** (2012), no. 2, 165–176.
29) **B. Balcerzak, A. Pierzchalski**, *Generalized gradients on Lie algebroids*, Ann. Global. Anal. Geom. **44**, no. 3 (2013) 319–337.
30) **B. Balcerzak, A. Pierzchalski**, *Derivatives of skew-symmetric and symmetric vector-valued tensors.* Works of The Mathematical Institute of the Ukrainian National Academy of Sciences. **10** (2013) pp. 35–55.
31) **B. Balcerzak, A. Pierzchalski**, *On Dirac operators on Lie algebroids.* Differential Geom. Appl. **35** (2014), suppl., 242–254.
32) **M. Ciska, A. Pierzchalski**, *On modulus of level sets of conjugate submersions.* Differential Geom. Appl. 36 (2014), 90–97.
33) **A. Kaźmierczak, A. Pierzchalski**, *On some criterion of conformality.* Balkan J. Geom. Appl. 21 (2016), no. 1, 51–57.
34) **A. Kimaczyńska, A. Pierzchalski**, *Elliptic operators in the bundle of symmetric tensors.* 50th Seminar "Sophus Lie", 193–218, Banach Center Publ., 113, Polish Acad. Sci. Inst. Math., Warsaw, 2017.
35) **A. Pierzchalski**, *Gradients: the ellipticity and the elliptic boundary conditions - a jigsaw puzzle* Folia Mathematica, Acta Universitatis Lodziensis Vol. 19, No. 1, pp. 65–83, 2017.