Ping-pong
and an entropy estimate in groups

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Abstract
We provide an entropy estimate from below for a finitely generated group of transformation of a compact metric space which contains a ping-pong game with players located anywhere in the group.

Introduction
The notion of entropy for finitely generated groups of transformations of compact metric spaces has been introduced (in the wider context of pseudogroups and foliations) by Ghys, Langevin and the second author [GLW] (see either [CC], Chapter 13, or [Wa] for more detailed expositions). It corresponds to the topological entropy of single transformations, depends on the choice of a generating set but its vanishing (or, non-vanishing) is independent of such a choice.

Ping-pong in transformation groups is attributed (see [Ha], Chapter II.B) to Feliks Klein who used it to study Kleinian groups. It implies some complexity of the dynamics, in particular, positive entropy and – in 1-dimensional dynamics – arises always when the dynamics of the system is complicated enough (see, for example, [SW] and the bibliography therein). In some sense, in one dimensional dynamics, ping-pong is related to horseshoes which can be used to estimate (or even, to calculate) entropies of the systems (see [LM] and, again, the bibliography therein).
It is known (see, for example, Prop. 2.4.10 in [Wa]) that ping-pong in a group implies the entropy estimate from below: entropy is greater or equal to the product of log 2 by the inverse of the maximum of distances (in the metric determined by a given generating set) of ping-pong players from the identity. Here, we produce a better estimate: we replace the denominator in the above by a quantity which arises from a well known lower bound for binomial distribution (see, for example, [Ash]) and is strictly larger that the quantity (maximal distance) of the “immediate” estimate mentioned above.

Our estimates can be adapted to pseudogroups and foliations to relate the value of entropy with the “strength” of a resilient orbit (or, of a resilient leaf) which can defined and related to the entropy and expressed in terms of the “length” of a piece of the orbit (or, of a leaf curve) providing ping-pong in the corresponding space (for foliations, via holonomy, on a transversal), see [LW]. A reader interested in such topics is referred also to Chapters 2 and 3 of [Wa].

1 Preliminaries

Throughout the paper, X will be a compact metric space with metric \(d\), \(G\) a finitely generated group of continuous transformations of \(X\) and \(G_1\) a fixed finite symmetric (i.e., such that \(e \in G_1\) and \(G^{-1} = \{g^{-1}; g \in G_1\} \subset G_1\)) set of generators for \(G\). For any \(n \in \mathbb{N}\), we put \(G_n = \{g_1 \circ \ldots \circ g_n; g_1, \ldots, g_n \in G_1\}\). Note that since \(e \in G_1\), \(G_1 \subset G_2 \subset G_3 \subset \ldots\).

**Definition 1 (Ping-pong).** Let \(G\) be a group acting on the compact metric space \(X\) and let \(A, A_1, A_2 \subset X\) be such that \(A_1, A_2 \subset A\) and \(\text{dist}(A_1, A_2) > 0\). We say that \(f_1, f_2 \in G\) are playing ping-pong if \(f_1(A) \subset A_1\) and \(f_2(A) \subset A_2\).

**Definition 2 (Entropy).** Let \(\varepsilon > 0\) and \(n > 0, n \in \mathbb{N}\). We say that points \(x, y \in X\) are \((n, \varepsilon)\)-separated if there exists a continuous map \(f \in G_n\) such that \(d(f(x), f(y)) \geq \varepsilon\).

A set \(A \subset X\) is \((n, \varepsilon)\)-separated if all the pairs of points \(x, y \in A, x \neq y\), have this property.

Since \(X\) is compact, every \((n, \varepsilon)\)-separated set is finite and we may put

\[
\sigma(n, \varepsilon, G_1) := \max\{\#A; A \subset X\text{ is } (n, \varepsilon)\text{-separated}\}
\]

and

\[
s(\varepsilon, G_1) := \limsup_{n \to \infty} \frac{1}{n} \log \sigma(n, \varepsilon, G_1).
\]
The number $h(G, G_1) := \lim_{\varepsilon \to 0} s(\varepsilon, G_1)$ is called the (topological) entropy of $G$ with respect to $G_1$.

For simplicity, in the sequel we will omit writing $G_1$ in all these formulas because we are interested in only one, fixed, set of generators.

In our calculations, we shall use the following (see, [Ash]) lower bound for the binomial distribution.

Lemma 3. Let $\Pr(\xi \leq k)$ be the probability that there are at most $k$ successes in $n$ trials with the probability $p$ of a single success. Then, if $p < \frac{k}{n} < 1$

$$Pr(\xi \leq k) \geq \frac{1}{\sqrt{2n}} \exp(-nD(\frac{k}{n}||p)),$$

where $D(a||p) = a \log \frac{a}{p} + (1 - a) \log \frac{1 - a}{1 - p}$.

Obviously,

Lemma 4. $\sum_{i=0}^{k} \binom{n}{i} = 2^n \cdot Pr(\xi \leq k) \geq 2^n \cdot \frac{1}{\sqrt{2n}} \exp(-nD(\frac{k}{n}||p))$.

2 Results

Theorem 5. Let $G$ be a group of transformations of $X$ containing two continuous maps $f_1, f_2$ playing ping-pong. If $f_1 \in G_{m_1}$, $f_2 \in G_{m_2}$ and $m_1 < m_2$, then the entropy $h(G)$ satisfies

$$h(G) \geq -q \log q - (1 - q) \log(1 - q)$$

$$\frac{qm_1}{qm_1 + (1 - q)m_2},$$

where $q$ is the solution of $q^{m_2/m_1} + q - 1 = 0$.

Proof. Let $X$ be a compact metric space. Take $f_1, f_2, A_1, A_2$ as in Definition 1 and $\varepsilon < \text{dist}(A_1 \cap A_2)$. Choose any $c \in X$. Define the set $E_{n,k} := \{f_{i_1} \circ \ldots \circ f_{i_n}(c); i_j \in \{1, 2\}, \# \{j : i_j = 1\} \geq k\}$. From $A_1 \cap A_2 = \emptyset$ we gain that points $f_{i_1} \circ \ldots \circ f_{i_n}(c)$ and $f_{j_1} \circ \ldots \circ f_{j_n}(c)$ are different when $\{i_1, \ldots, i_n\} \neq \{j_1, \ldots, j_n\}$.

Therefore, we obtain the inequality $\#E_{n,k} \geq \sum_{i=0}^{n-k} \binom{n}{i}$.

Moreover, if $x = f_{i_1} \circ \ldots \circ f_{i_m}(c)$, $y = f_{j_1} \circ \ldots \circ f_{j_n}(c)$ are different points of $E_{n,k}$, then $d((f_{i_1} \circ \ldots \circ f_{i_m})^{-1}(x), (f_{j_1} \circ \ldots \circ f_{j_m})^{-1}(y)) \geq \varepsilon$ where $m$ is the largest number satisfying the condition $i_1 = j_1$, $\ldots$, $i_m = j_m$. 

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Furthermore, $(f_1 \circ \ldots \circ f_m)^{-1} \in G_{m_1k + m_2(n-k)}$. Thus the set $E_{n,k}$ is $(km_1 + (n-k)m_2, \varepsilon)$-separated and $s(km_1 + (n-k)m_2, \varepsilon) \geq \sum_{j=0}^{n-k} \binom{n}{j}$.

Notice that for any $n \in \mathbb{N}$ number $k$ must be smaller or equal to $n$ and that is why we will use $(k_n)$ – a sequence of numbers from $\{0, \ldots, n\}$.

For any sequence like that we obtain the estimate
\[
h(G) \geq \lim_{n \to \infty} \frac{1}{k_nm_1 + (n-k)m_2} \log \sum_{i=0}^{n-k_n} \binom{n}{i}
\]

By Lemma 3 and 4
\[
\sum_{i=0}^{n-k_n} \binom{n}{i} \geq 2^n \cdot \frac{1}{\sqrt{2n}} \exp(-n\left(\frac{k_n}{n} \log \frac{k_n}{n} + \frac{n-k_n}{n} \log \frac{n-k_n}{n} + \log 2\right))
\]
and then putting $k_n = qn$ for a given $q < 1$ we obtain the inequality
\[
h(G) \geq \frac{-q \log q - (1-q) \log(1-q)}{qm_1 + (1-q)m_2} =: \phi(q).
\]

The best result is obtained for the maximal value of $\phi(q)$. Of course, $\lim_{q \to 0^+} \phi(q) = 0$, and $\lim_{q \to 1^-} \phi(q) = 0$. Also, $\phi(q)$ is positive for all $q \in (0,1)$, thus our function $q \mapsto \phi(q)$ must attain a maximal value in the interval $(0,1)$.

By computing the derivative $\phi'$ and writing the equation $\phi'(q) = 0$ we obtain that the maximal value of $\phi$ is attained at $q$ which satisfies $m_1 \log(1-q) - m_2 \log q = 0$, equivalently $q^{m_2/m_1} + q - 1 = 0$.

For ping-pong players in the generating set, the Theorem yields directly the “immediate” estimate mentioned in Introduction:

**Corollary 6.** For $f_1, f_2 \in G_1$, that is for $m_1 = m_2 = 1$ we have
\[
h(G) \geq \log 2 \approx 0.6931471806.
\]

**Example 7.** Computer aided numerical calculations yield the following estimates:

1. for $(m_1; m_2) = (1; 3)$:
   $q \approx 0.6823296719378$ and $h(G) \geq 0.38224505835113 > \frac{1}{3} \log 2$,

2. for $(m_1; m_2) = (2; 5)$:
   $q \approx 0.654050569930569$ and $h(G) \geq 0.212289420476104 > \frac{1}{5} \log 2$,.
3. for \((m_1, m_2) = (1, 7)\):
\[q \approx 0, 796543459630383\] and \(h(G) \geq 0, 22747246488779 > \frac{1}{7} \log 2\)
and so on.

References


